Perron Formula Without Error Estimates

The Perron formula for Dirichlet series plays the same rôle as the Cauchy formula for power series. Consider the Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

Let σ_0 be its abscissa of convergence and and $\bar{\sigma}$ be its abscissa of absolute convergence.

Perron Formula. If x is not an integer and c is a positive number and $\sigma>\sigma_0-c$, then

$$\sum_{n<x}\frac{a_n}{n^s}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s+w)\frac{x^w}{w}dw.$$

Remark. The domain of convergence of a power series is a disk whereas the domain of convergence of a Dirichlet series is a left half-plane. So the integration in the Cauchy formula for a power series is a circle (which is the boundary of a disk) whereas the integration in the Perron formula is a vertical line (which is the boundary of a half-plane). There is a difference. The Perron formula gives a partial sum of the Dirichlet series instead of the whole series. It corresponds to the following truncated form of the Cauchy formula.

$$\begin{aligned}\sum_{n=1}^{m}a_n\left(z-c\right)^n&=\frac{1}{2\pi i}\int_{|w-c|=r}f(w)\left(\sum_{n=1}^{m}\frac{(z-c)^n}{(w-c)^{n+1}}\right)dw\\&=\frac{1}{2\pi i}\int_{|w-c|=r}f(w)\frac{1}{w-c}\frac{1-\left(\frac{z-c}{w-c}\right)^{m+1}}{1-\frac{z-c}{w-c}}\,dw\\&=\frac{1}{2\pi i}\int_{|w-c|=r}f(w)\frac{1-\left(\frac{z-c}{w-c}\right)^{m+1}}{w-z}\,dw.\end{aligned}$$

The usual Cauchy formula corresponds to the case $m=\infty$.

Lemma.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w}$$

is equal to 1 if $n<x$ and is equal to 0 if $n>x$.

Proof of Lemma. Let $a = \frac{x}{n}$. The identity can be rewritten as follows.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}\,dz$$

is equal to 1 if $a > 1$ and is equal to 0 if $0 < a < 1$. We verify this in the two cases separately.

Assume first $a > 1$. Then $\log a > 0$ and we do the integration from $c - iR$ to $c+iR$ and then the left circle C_R whose diameter is on $[c - iR, c + iR]$. The integral along C_R goes to zero as $R \to \infty$ for the following reason. Given any $\varepsilon > 0$ there exists some $A = A_\varepsilon$ such that the integral over $C_R \cap \{\operatorname{Re} z < -A\}$ is less than ε, because $|a^z| < e^{-A\log a}$ on $\{\operatorname{Re} z < -A\}$ and, for fixed A, the expression $\left|\frac{1}{z}\right| \to 0$ on $C_R \cap \{\operatorname{Re} z < -A\}$ as $R \to \infty$ while the integral of $|a^z\,dz|$ on $C_R \cap \{\operatorname{Re} z < -A\}$ remains bounded as $R \to \infty$. Finally the pole of $\frac{a^z}{z}$ at $z = 0$ contributes the residue 1. Thus the integral we seek is 1.

Assume now $a < 1$. Then $\log a < 0$ and we use the right half-circle whose diameter is on $[c - iR, c + iR]$. In this case there is no pole inside the contour and the integral we seek is 0. Q.E.D.

Remark. This lemma corresponds to

$$\frac{1}{2\pi i}\int_{|z|=r} z^n\,dz$$

equal to 1 if $n = -1$ and equal to 0 if $n \neq -1$, in the case of power series and is used to pick out one coefficient in a power series.

Proof of the Perron Formula. Suppose first $\sigma > \bar\sigma - c$. Absolute and uniform convergence gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iU}^{c+iT} f(s+w)\frac{x^w}{w}dw &= \frac{1}{2\pi i}\int_{c-iU}^{c+iT}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\frac{x^w}{w}\,dw\\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-iU}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w}.\end{aligned}$$

By the Lemma, we need only show that

$$\lim_{T\to\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c+iT}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w} = 0$$

and
$$\lim_{U\to\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-i\infty}^{c-iU}\left(\frac{x}{n}\right)^w\frac{dw}{w}=0.$$
For fixed x, integration by parts gives
$$\int_{c+iT}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w}=-\left(\frac{x}{n}\right)^{c+iT}\frac{1}{\left(\log\frac{x}{n}\right)(c+iT)}+\frac{1}{\log\frac{x}{n}}\int_{c+iT}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w^2}$$
$$=O\left(\frac{1}{n^cT}\right)+O\left(\frac{1}{n^c}\int_T^{\infty}\frac{dv}{c^2+v^2}\right)=O\left(\frac{1}{n^cT}\right).$$
Hence
$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c+iT}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w}=O\left(\frac{1}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}\right)$$
and likewise
$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-i\infty}^{c-iU}\left(\frac{x}{n}\right)^w\frac{dw}{w}=O\left(\frac{1}{U}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}\right).$$
This finishes the proof of the case $\sigma>\bar\sigma-c$.

For the case of $\sigma_0-c<\sigma\le\bar\sigma-c$ we use the growth rate of the Dirichlet series on the vertical strip bounded by the abscissa of convergence and the abscissa of absolute convergence. Take $\alpha>\bar\sigma-c$ and use the contour integration of
$$f(s+w)\left(\frac{x}{n}\right)^w\frac{dw}{w}$$
over the rectangle
$$\left\{c\le\operatorname{Re}w\le\alpha.\ -U\le\operatorname{Im}w\le T\right\}.$$
By
$$f(s)=O\left(t^{-(\sigma+c-\sigma_0)+\varepsilon}\right),$$
the integrals over the two horizontal line-segments approach 0 as U, $T\to\infty$. Hence
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s+w)\frac{x^w}{w}dw=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}f(s+w)\frac{x^w}{w}dw$$
and we are reduced to the previous case of $\sigma>\bar\sigma-\alpha$. This finishes the proof of the Perron formula. The case of $s=0$ gives us the following.

Special Case of Perron Formula.

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w} dw \quad \text{for } c > \sigma_0.$$