Math 213a Homework September 25, 2004

Problem 1 (Stein, p.27, #8). Suppose U and V are open sets in the complex plane $\mathbb{C}$. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z},$$
$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

This is the complex version of the chain rule. Recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

when $z = x + \sqrt{-1}\,y$.

Problem 2 (Stein, p.27, #9). Show that in polar coordinates $z = re^{\sqrt{-1}\,\theta}$, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + \sqrt{-1}\,\theta \quad \text{where } z = re^{\sqrt{-1}\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Problem 3 (Stein, p.27, #10 & #11). Verify that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Prove that if f is holomorphic in the open set Ω, then the real and imaginary part of f are harmonic in the sense that their Laplacians are zero.

Problem 4 (Stein, p.27, #12). Consider the function defined by

$$f\left(x+\sqrt{-1}\,y\right) = \sqrt{|x|\,|y|}, \quad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f fails to be holomorphic at 0 (in the sense that the complex derivative of f exists at every point in some neighborhood of 0).

Problem 5 (Stein, p.28, #13). Suppose f is holomorphic in a connected open set Ω. Prove that in any one of the following cases.

(i) $\operatorname{Re} f$ is constant;

(ii) $\operatorname{Im} f$ is constant;

(iii) $|f|$ is constant;

one can conclude that f is constant.

Problem 6 (Stein, p.31, #25(c)). Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi\sqrt{-1}}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive (*i.e.*, counter-clockwise) orientation. (*Hint:* apply Cauchy's integral formula.)

Problem 7 (Stein, p.66, #11). Let f be a holomorphic function on the disk D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(Re^{\sqrt{-1}\,\varphi}\right) \operatorname{Re}\left(\frac{Re^{\sqrt{-1}\varphi}+z}{Re^{\sqrt{-1}\varphi}-z}\right) d\varphi.$$

(b) Show that

$$\operatorname{Re}\left(\frac{Re^{\sqrt{-1}\gamma}+z}{Re^{\sqrt{-1}\gamma}-z}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

Hint: For the first part, note that if

$$w = \frac{R^2}{\bar{z}},$$

then the integral of

$$\frac{f(\zeta)}{\zeta - w}$$

around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.

Problem 8 (Stein, p.66, #12). Let u be a real-valued function defined on the unit disk $\mathfrak{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\Delta u(x, y) = 0$$

for all $(x, y) \in \mathfrak{D}$.

(a) Prove that there exists a holomorphic function f on the unit disk such that

$$\operatorname{Re} f = u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. (Hint: From the Cauchy-Riemann equations $f'(z) = 2\,\frac{\partial u}{\partial z}$. Therefore, let $g(z) = 2\,\frac{\partial u}{\partial z}$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\operatorname{Re} F$ differs from u by a real constant.)

(b) Deduce from this result and from Problem 6 the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disk and continuous on its closure, then if $z = r\,e^{\sqrt{-1}\,\theta}$ one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r\left(\theta - \varphi\right) u(\varphi) d\varphi,$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disk given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$