Midterm Problems

Problem 1

Let

```
In[1]:= A = (  1   2 )
            ( -1  -1 )
        B = ( 3  0 )
            ( 2  1 )

Out[1]= {{1, 2}, {-1, -1}}

Out[2]= {{3, 0}, {2, 1}}
```

Find

```
In[6]:= 3 A + 2 B // MatrixForm
        A .B // MatrixForm
        A. A. Transpose[B] // MatrixForm

Out[6]//MatrixForm=
        ( 9   6 )
        ( 1  -1 )

Out[7]//MatrixForm=
        (  7   2 )
        ( -5  -1 )

Out[8]//MatrixForm=
        ( -3  -2 )
        (  0  -1 )
```

Problem 2

Find the parametric form to the general solution to the system of linear equations

$$2x_1 + 4x_2 + x_4 = 2;$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4;$$
$$3x_1 + 6x_2 + 2x_3 + x_4 = 4;$$

Solution. The augmented matrix of this system of linear equations is

```
In[11]:= M2 = ( 2  4  0  1  2 )
              ( 1  2  3  2  4 )
              ( 3  6  2  1  2 )

Out[11]= {{2, 4, 0, 1, 2}, {1, 2, 3, 2, 4}, {3, 6, 2, 1, 2}}
```

which has reduced row echelon form

```
In[12]:= RowReduce[M2] // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

This means the general solution takes the form

$x_4 = 2;$
$x_3 = 0$
x_2 is free
$x_1 = -2\, x_2$

Another way to write this is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\, x_2 \\ z_2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} + z_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which is the parametric form

Problem 3

Determine whether the vectors

$$\text{In[14]:= } v1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 4 \end{pmatrix};\ v2 = \begin{pmatrix} 3 \\ 7 \\ -14 \\ 8 \end{pmatrix};\ v3 = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 2 \end{pmatrix};$$

are linearly independent.

Solution. As we know, the vectors are linearly independent if and only if the equation $\mathbf{v}_1\, x_1 + \mathbf{v}_2\, x_2 + \mathbf{v}_3\, x_3 = \mathbf{0}$ has a nontrivial solution. This will be determined by the (reduced) row echelon form of the matrix $(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4)$.

```
In[25]:= M3 = Transpose[{Flatten[v1], Flatten[v2], Flatten[v3]}]

Out[25]= {{1, 3, 0}, {3, 7, 1}, {-2, -14, 4}, {4, 8, 2}}

In[27]:= RowReduce[M3] // MatrixForm
```

Out[27]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Apparently, then, the vector equation $\mathbf{v}_1\, x_1 + \mathbf{v}_2\, x_2 + \mathbf{v}_3\, x_3 = \mathbf{0}$ has a free variable, x_3. So the set of vectors is linearly *dependent*.

Problem 4

Let $T:\mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates counterclockwise by 30° then reflects in the x_2-axis. What is the matrix associated to this linear transformation?

Solution (Graphics to follow). When the standard basis vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is rotated counterclockwise by 30°, it becomes the vector $\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$. After flipping this in the x_2-axis, we have $\begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}$.
For the other standard basis vector $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it first goes to $\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$ and then to $\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$. So the matrix is

In[28]:= $\mathbf{M4} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

Out[28]= $\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}\right\}\right\}$

Problem 5

True or false:

(a) Any system of n linear equations in n variables has at most n solutions.

Solution. This is **false**. The system may have infinitely many solutions if there is a free variable.

(b) If the system $A\mathbf{x}=\mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x}=\mathbf{0}$.

Solution. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to the equation; that is $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$. Then $A(\boldsymbol{x}_1 - \mathbf{x}_2) = \mathbf{0}$. This along with the trivial solution 0 gives us at least two solutions to the system $A\mathbf{x}=\mathbf{0}$. So the statement is **true.**

(c) If *A, B,* and *C* are matrices, *AB*=*C* and *C* has 3 columns, then *A* has three columns.

Solution. Suppose *A* has dimensions $m\times n$, *B* has dimensions $n \times p$. Then since *C* has three columns, $p = 3$. But it is not necessarily true that *A* has three columns unless *B* has three rows. For instance:
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{pmatrix} \begin{pmatrix} \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{8} & \mathbf{9} & \mathbf{10} \end{pmatrix} = \begin{pmatrix} \mathbf{21} & \mathbf{24} & \mathbf{27} \\ \mathbf{47} & \mathbf{54} & \mathbf{61} \end{pmatrix}$
The statement is **false**.

(d)If *A* is square and has a pivot position in every row, then there exists a matrix *B* such that $AB = I$.

Solution. This statement is **true.** For if *A* has a pivot position in every row and is square, then it has a pivot position in every column. Therefore by our big theorem of invertibility, *A* is invertible. Thus any of the equivalent criteria for invertibility hold, in particular, there is a right inverse *B* for *A*.

Problem 6

Let A be an $n \times n$ matrix that is not invertible. Show there exists a (nonzero) $n \times n$ matrix B such that AB is the $n \times n$ matrix of zeroes.

Solution. Since A is not invertible, there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v}=\mathbf{0}$. Let B be the matrix $(\mathbf{v} \;\; \mathbf{v} \;\; \ldots \;\; \mathbf{v})$. Then $AB=O$.