The Assignment Problem

Math 20

Linear Algebra and Multivariable Calculus

October 13, 2004

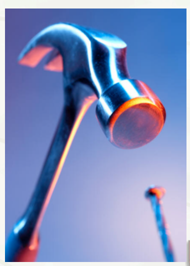
The Problem

- Three air conditioners need to be installed in the same week by three different companies
- Bids for each job are solicited from each company
- To which company should each job be assigned?

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Naïve Solution

- There are only 6 possible assignments of companies to jobs
- Check them and compare



Naïve Solution—Guess #1

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 53 + 87+ 36 = 176

Naïve Solution—Guess #2

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 53 + 92+ 41 = 186

Naïve Solution—Guess #3

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 47 + 96 + 36 = 179

Naïve Solution—Guess #4

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 47 + 92 + 37 = 176

Naïve Solution—Guess #5

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 47 + 96 + 41 = 197

Naïve Solution—Guess #6

	Bldg1	Bldg2	Bldg3
A	53	96	37
B	47	87	41
C	60	92	36

Total Cost = 60 + 87 + 37 = 184

Naïve Solution—Completion

	1	2	3
A	53	96	37
B	47	87	41
C	60	92	36

	1	2	3
A	53	96	37
B	47	87	41
C	60	92	36

	1	2	3
A	53	96	37
B	47	87	41
C	60	92	36

	1	2	3
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	1	2	3
A	53	96	37
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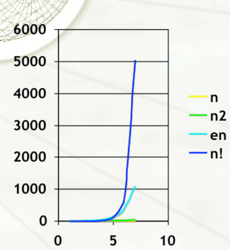
C	60	92	36

Disadvantages of Naïve Solution



- How does the time-to-solution vary with problem size?
- Answer: O(n!)

Rates of Growth

n	log(n)	n	n^2	e^n	n!
1	0.00000	1	1	2.7	1
2	0.30103	2	4	7.4	2
3	0.47712	3	9	20.1	6
4	0.60206	4	16	54.6	24
5	0.69897	5	25	148.4	120
6	0.77815	6	36	403.4	720
7	0.84510	7	49	1096.6	5040
8	0.90309	8	64	2981.0	40320
9	0.95424	9	81	8103.1	362880
10	1.00000	10	100	22026.5	3628800
11	1.04139	11	121	59874.1	39916800
12	1.07918	12	144	162754.8	479001600
13	1.11394	13	169	442413.4	6227020800
14	1.14613	14	196	1202604.3	8.7178E+10
15	1.17609	15	225	3269017.4	1.3077E+12

Mathematical Modeling of the Problem

- Given a *Cost Matrix* C which lists for each "company" *i* the "cost" of doing "job" *j*.
- Solution is a *permutation matrix* X : all zeros except for one 1 in each row and column
- Objective is to minimize the total cost

$$c_{\text{total}} = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

An Ideal Cost Matrix

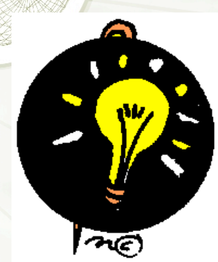
- All nonnegative entries
- An possible assignment of zeroes, one in each row and column
- In this case the minimal cost is apparently zero!

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 0 & 0 \end{bmatrix}$$

The Hungarian Algorithm

- Find an "ideal" cost matrix that has the same optimal assignment as the given cost matrix
- From there the solution is easy!

Critical Observation



- Let C be a given cost matrix and consider a new cost matrix C' that has the same number added to each entry of a single row of C
- For each assignment, the new total cost differs by that constant
- The optimal assignment is the same as before

Critical Observation

- Same is true of columns
- So: we can subtract *minimum* entry from each row and column to insure nonnegative entries

On our given matrix

$$\begin{bmatrix} 53 & 96 & 37 \\ 47 & 87 & 41 \\ 60 & 92 & 36 \end{bmatrix} \mapsto \begin{bmatrix} 16 & 59 & 0 \\ 6 & 46 & 0 \\ 24 & 56 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

Still Not Done

- No assignment of zeros in this matrix

$$\begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

Still Not Done

- No assignment of zeros in this matrix

$$\begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

Still Not Done

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Still Not Done

- No assignment of zeros in this matrix

$$\begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

?

Still Not Done

- No assignment of zeros in this matrix
- Still, we can create new zeroes by subtracting the smallest entry from some rows

$$\begin{bmatrix} 0 & 3 & -10 \\ 0 & 0 & 0 \\ 8 & 0 & -10 \end{bmatrix}$$

Still Not Done

- No assignment of zeros in this matrix
- Still, we can create new zeroes by subtracting the smallest entry from some rows
- Now we can preserve nonnegativity by adding that entry to columns which have negative entries

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 0 & 0 \end{bmatrix}$$

Solutions

$$\begin{bmatrix} \textcircled{0} & 3 & 0 \\ 0 & \textcircled{0} & 10 \\ 8 & 0 & \textcircled{0} \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & \textcircled{0} \\ \textcircled{0} & 0 & 10 \\ 8 & \textcircled{0} & 0 \end{bmatrix}$$

Naïve Solution—Completion

	1	2	3
A	53	96	37
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C	60	92	36

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	1	2	3
A	53	96	37
B	47	87	41
C	60	92	36

The Hungarian Algorithm

1. Find the minimum entry in each row and subtract it from each row
2. Find the minimum entry in each column and subtract it from each column

- Resulting matrix is nonnegative

The Hungarian Algorithm

3. Using lines that go all the way across or all the way up-and-down, cross out all zeros in the new cost matrix

- Find a way to do this with a minimum number of lines (≤n)

$$\begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

The Hungarian Algorithm

4. If you can only do this with n lines, an assignment of zeroes is possible.

$$\begin{bmatrix} 10 & 13 & 0 \\ 0 & 0 & 0 \\ 18 & 10 & 0 \end{bmatrix}$$

The Hungarian Algorithm

5. Otherwise, determine the smallest entry not covered by any line.
   - Subtract this entry from all uncovered entries
   - Add it to all double-covered entries
   - Return to Step 3.

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 3 & 0 \end{bmatrix}$$

The Hungarian Algorithm

3. Using lines that go all the way across or all the way up-and-down, cross out all zeros in the new cost matrix

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 3 & 0 \end{bmatrix}$$

The Hungarian Algorithm

3. Using lines that go all the way across or all the way up-and-down, cross out all zeros in the new cost matrix

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 3 & 0 \end{bmatrix}$$

The Hungarian Algorithm

3. Using lines that go all the way across or all the way up-and-down, cross out all zeros in the new cost matrix

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 3 & 0 \end{bmatrix}$$

The Hungarian Algorithm

4. If you can only do this with n lines, an assignment of zeroes is possible.

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 3 & 0 \end{bmatrix}$$

Solutions

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 10 \\ 8 & 0 & 0 \end{bmatrix}$$

Example 2

- A cab company gets four calls from four customers simultaneously
- Four cabs are out in the field at varying distance from each customer
- Which cab should be sent where to minimize total (or average) waiting time?

		Customer			
		1	2	3	4
Cab	A	9	7.5	7.5	8
	B	3.5	8.5	5.5	6.5
	C	12.5	9.5	9.0	10.5
	D	4.5	11.0	9.5	11.5

Integerizing the Matrix

$$\begin{bmatrix} 9 & 7.5 & 7.5 & 8 \\ 3.5 & 8.5 & 5.5 & 6.5 \\ 12.5 & 9.5 & 9 & 10.5 \\ 4.5 & 10 & 9.5 & 11.5 \end{bmatrix} \mapsto \begin{bmatrix} 90 & 75 & 75 & 80 \\ 35 & 85 & 55 & 65 \\ 125 & 95 & 90 & 105 \\ 45 & 10 & 95 & 115 \end{bmatrix}$$

Non-negativizing the Matrix

$$\begin{bmatrix} 90 & 75 & 75 & 80 \\ 35 & 85 & 55 & 65 \\ 125 & 95 & 90 & 105 \\ 45 & 10 & 95 & 115 \end{bmatrix} \mapsto \begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix} \mapsto \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 25 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Covering the Zeroes

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 25 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Can do it with three!

Find Smallest Uncovered Entry

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & \textcircled{20} & 25 \\ 25 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Subtract and Add

$$\begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 30 & \textcircled{0} & 5 \\ 55 & 5 & 0 & 10 \\ 0 & 45 & 30 & 45 \end{bmatrix}$$

Cover Again

$$\begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 30 & 0 & 5 \\ 55 & 5 & 0 & 10 \\ 0 & 45 & 30 & 45 \end{bmatrix}$$

Still three!

Find Smallest Uncovered

$$\begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 30 & 0 & 5 \\ 55 & 5 & 0 & 10 \\ 0 & 45 & 30 & 45 \end{bmatrix}$$

Subtract and Add

$$\begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Cover Again

$$\begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Done!

Solutions

$$\begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix} \quad \begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Done!

Solutions

$$\begin{bmatrix} 9 & 7.5 & 7.5 & 8 \\ 3.5 & 8.5 & 5.5 & 6.5 \\ 12.5 & 9.5 & 9 & 10.5 \\ 4.5 & 10 & 9.5 & 11.5 \end{bmatrix} \quad \begin{bmatrix} 9 & 7.5 & 7.5 & 8 \\ 3.5 & 8.5 & 5.5 & 6.5 \\ 12.5 & 9.5 & 9 & 10.5 \\ 4.5 & 10 & 9.5 & 11.5 \end{bmatrix}$$

Total Cost = 27.5

Other Applications of AP

- Assigning teaching fellows to time slots
- Assigning airplanes to flights
- Assigning project members to tasks
- Determining positions on a team
- Assigning brides to grooms (once called the *marriage problem*)