MATHEMATICS 1b
Final Examination
January 24, 1996

1. Calculate the following integrals:

a) $\int \sec^2\theta \, d\theta$ $\quad \left[\text{Note} : \sec\theta = \dfrac{1}{\cos\theta}\right]$

b) $\int z \, e^{3z^2+1} \, dz$

c) $\int y \sec^2 y \, dy$

d) $\int \dfrac{dt}{1+\sqrt{t}}$

e) $\int \dfrac{dx}{(b+ax)^2}$ $\quad a, b$ constants

2. A lake contains pollutants. A stream feeds clear mountain water into the lake at 2 gals/min. Polluted water is drained out of the lake at a rate of 2 gals/min by a second stream. If the volume of the lake is Vgals and time, t, is measured in minutes, and if it is assumed that the pollutants are spread evenly through the lake at all times, then the differential equation for $Q(t)$, the quantity of pollutant in the lake at time t, is (circle one):

a) $\dfrac{dQ}{dt} = -2Q$

b) $\dfrac{dQ}{dt} = -2t$

c) $\dfrac{dQ}{dt} = -\dfrac{2Q}{V}$

d) $\dfrac{dQ}{dt} = 2 - 2Q$

e) $\dfrac{dQ}{dt} = 2 - \dfrac{2Q}{V}$

f) $\dfrac{dQ}{dt} = 2V - Qt$

g) $\dfrac{dQ}{dt} = 2Qt - V$

h) $\dfrac{dQ}{dt} = Qe^{-2t}$

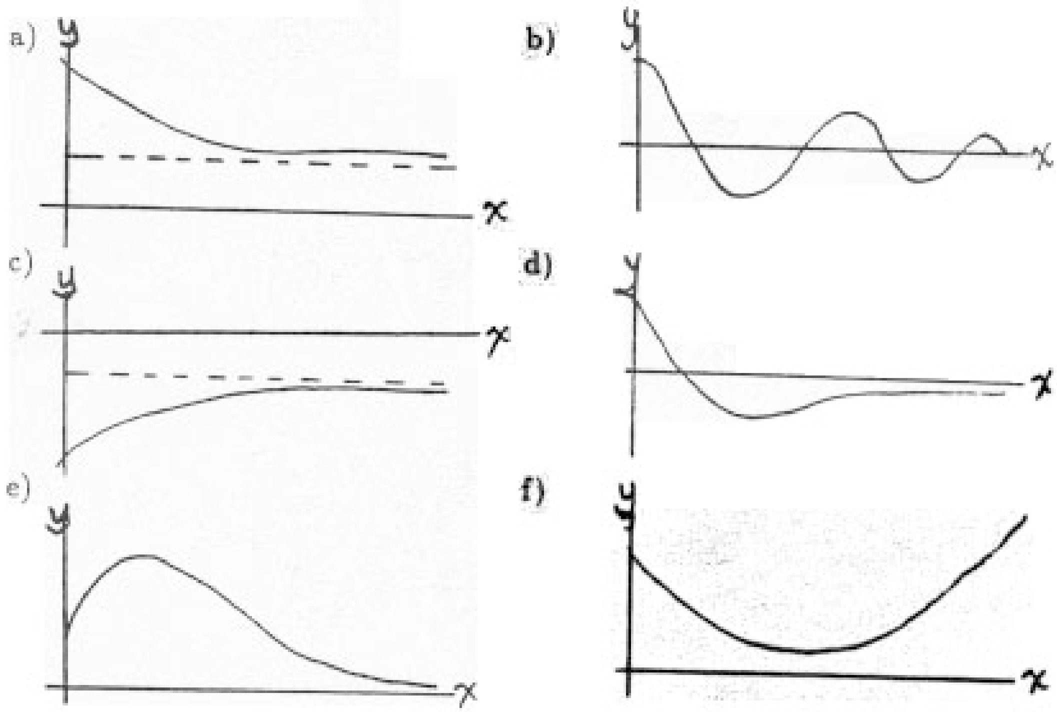
i) $\dfrac{dQ}{dt} = \dfrac{Q}{V}e^{-2t}$

j) $\dfrac{dQ}{dt} = \dfrac{2Q-2V}{V}$

3. Suppose y is a solution to the differential equation

$$\frac{dy}{dx} = f(y)$$

and that $f(y) \geq 0$ for all y. Which of the following could be a graph of y? (Circle one or more)



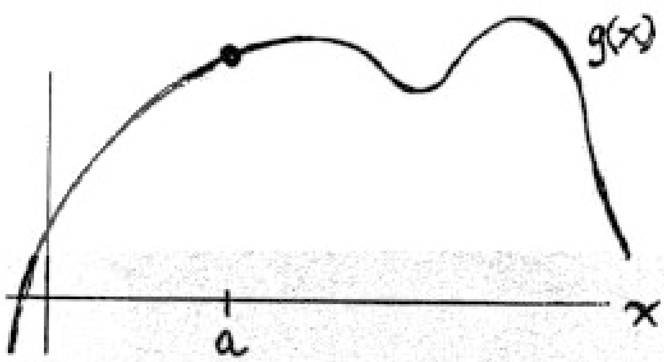
4. Write down the fourth degree Taylor polynomial for $\cos(3x^2)$ about $x = 0$.

5. Suppose a function satisfies $f(2) = 4$, $f'(2) = 3$, $f''(2) = -5$, $f'''(2) = 12$. Write down the third degree Taylor polynomial for f about $x = 2$.

6. The function g has the Taylor approximation

$$g(x) \approx c_0 + c_1(x - a) + c_2(x - a)^2.$$

and the graph given below:

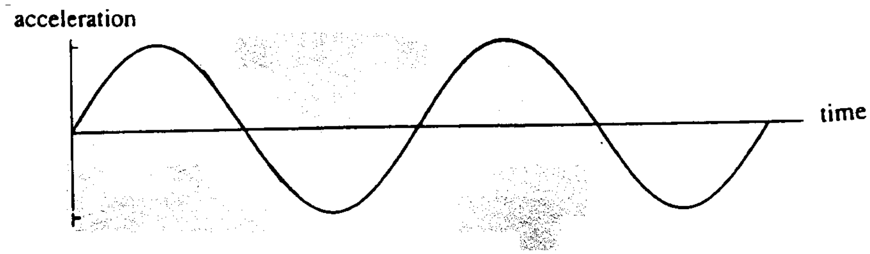


What can you say about the signs of c_0, c_1, c_2? (Circle your answers; no reasons need be given.)

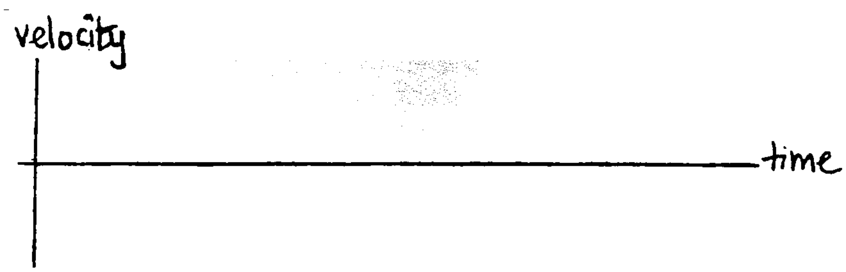
a) c_0 is negative   zero   positive
b) c_1 is negative   zero   positive
c) c_2 is negative   positive

7. *Acceleration* is the rate of change of velocity. A positive acceleration indicates an increasing velocity and a negative acceleration indicates a decreasing velocity.

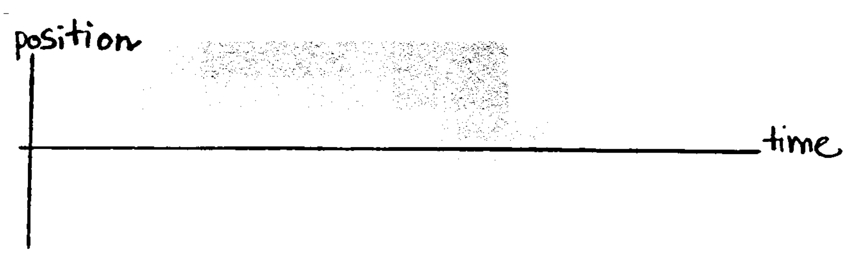
An ion (a charged particle) is at rest on the x-axis at $x = 0$ at time $t = 0$. Subsequently the particle moves as a result of an electric field and undergoes both positive and negative acceleration as shown in the figure. Suppose positive velocity represents motion in the positive x-direction and negative velocity represents motion in the negative x-direction.



a) Draw a graph of the velocity of the particle as a function of time.



b) Draw a graph of the position of the particle as a function of time.



8. Suppose the government spends \$1 million on highways. Some of this money is earned by the highway workers who in turn spend \$500,000 on food, travel, entertainment.

This causes \$250,000 to be spend by the workers in the food, travel, and entertainment industries. This \$250,000 causes another \$125,000 to be spent; the \$125,000 causes another \$62,500 to be spent, and so on. (Notice that each expenditure is half the previous one.) Assuming this process continues forever, what is the total spending generated by the original \$1 million expenditure? (Include the original \$1 million in your total.)

9. At time t (measured in hours), $0 \leq t \leq 24$, a firm uses electricity at a rate of $E(t)$ kilowatts. The cost of power at time t is $C(t)$ dollars per kilowatt-hour. Assume that

$$E(t) = 5 + \sin\left(\frac{\pi t}{12}\right) \qquad \text{and} \qquad C(t) = 0.1 + 0.01\sin\left(\frac{\pi t}{24}\right)$$

a) Set up a Riemann sum which approximates the total cost of electricity consumed by the firm over a 24-hour period.

b) Give a definite integral which represents the total cost of the electricity consumed by the firm over a 24-hour period. (Do not evaluate the integral.)

10.

a) Find a formula for the solution to the differential equation

$$\frac{dy}{dx} = y\cos x \qquad\qquad y(0) = 4.$$

b) Find the exact minimum and maximum values of the function $y(x)$ you found in part a).

11. Suppose x and y represent the populations (in millions) of two species X and Y of animals, and that x and y satisfy the differential equations

$$\frac{dx}{dt} = 6x - 2x^2 - xy \qquad\qquad \frac{dy}{dt} = 4y - 2y^2 - xy$$

a) On the axes below, draw a phase plane diagram. Label each nullcline with its equation and the coordinates of the x- and y-intercepts. Show the general direction of the trajectories in each region.

b) What are the coordinates of the equilibrium points?

c) What type of interactions do these species have? (Circle one; no reasons needed)

i)	They are competitors	iii)	Species X preys on species Y
ii)	They help one another	iv)	Species Y preys on species X

d) If $x(0) > 0$ and $y(0) > 0$, what happens to the population in the long run? Does this behavior depend on the specific values of the initial populations?

12. Consider the differential equation

$$2\frac{d^2y}{dt^2} + c\frac{dy}{dt} + 4y = 0$$

where $c \geq 0$ is a constant.

a) Assume $c = 6$. Find a formula for the general solution $y(t)$, and draw a graph of a possible nonzero solution.

b) Assume $c = 5$. Find a formula for the general solution, $y(t)$.

c) For what values of c does the solution, $y(t)$, represent an oscillation (that is, a solution which crosses the t-axis infinitely many times)?

d) Suppose that the value of c is one of the values found in part c) making the solution represent an oscillation.

(i) Express the period of the oscillation as a function of c.

(ii) What is the minimum value of the period of the oscillation? Give the exact answer.

(iii) What value of c make the period of the oscillation 10? Give your answer to 2 decimal places.