1. $y = x - x^{-1} \Rightarrow y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\text{LHS} = xy' + y = x\left(1 + x^{-2}\right) + \left(x - x^{-1}\right) = x + x^{-1} + x - x^{-1} = 2x = \text{RHS}$$

2. $y = \sin x \, \cos x - \cos x \Rightarrow y' = \sin x \, (-\sin x) + \cos x \, (\cos x) - (-\sin x) = \cos^2 x - \sin^2 x + \sin x$.

$$\text{LHS} = y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \, \cos x - \cos x)$$

$$= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x = \text{RHS},$$

so y is a solution of the differential equation. Also, $y(0) = \sin 0 \, \cos 0 - \cos 0 = 0 \cdot 1 - 1 = -1$, so the initial condition is satisfied.

4. (a) $y = \cos kt \Rightarrow y' = -k \sin kt \Rightarrow y'' = -k^2 \cos kt$. Substituting these expressions into the differential equation $4y'' = -25y$, we get $4(-k^2 \cos kt) = -25(\cos kt) \Rightarrow (25 - 4k^2) \cos kt = 0$ [for all t] $\Rightarrow 25 - 4k^2 = 0 \Rightarrow k^2 = \frac{25}{4} \Rightarrow k = \pm\frac{5}{2}$.

(b) $y = A \sin kt + B \cos kt \Rightarrow y' = Ak \cos kt - Bk \sin kt \Rightarrow y'' = -Ak^2 \sin kt - Bk^2 \cos kt$. The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\text{LHS} = 4y'' + 25y = 4(-Ak^2 \sin kt - Bk^2 \cos kt) + 25(A \sin kt + B \cos kt)$$

$$= -4Ak^2 \sin kt - 4Bk^2 \cos kt + 25A \sin kt + 25B \cos kt$$

$$= (25 - 4k^2)A \sin kt + (25 - 4k^2)B \cos kt$$

$$= 0 \quad \text{since } k^2 = \frac{25}{4}.$$

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \Rightarrow 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \dfrac{P}{4200} < 1 \Rightarrow P < 4200 \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\dfrac{dP}{dt} = 0 \Rightarrow P = 4200 \text{ or } P = 0$

12. The graph for this exercise is shown in the figure at the right.

   A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can

      see that $y' < 0$ for some points in the first quadrant.

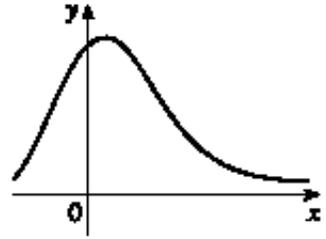   B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$.

   Thus, equations A and B are incorrect, so the correct equation is C.

   C. $y' = 1 - 2xy$ seems reasonable since:

      (1) When $x = 0$, y' could be 1.

      (2) When $x < 0$, y' could be greater than 1.

      (3) Solving $y' = 1 - 2xy$ for y gives us $y = \dfrac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \to \infty$, $y \to 0^+$, as

         shown in the figure.

Problem Set 22

Differential Equations Supplement
Problems: 1, 2, 4 on pages 990-991

April 20, 2006

Problem 1

Part A

We have an annual interest rate of 4% per year, compounded continuously. Turning 4% into a decimal yields .04 and since it is compounded continuously we have a relation for the rate of change of money of the form:

$$\frac{dM(t)}{dt} = .04M(t)$$

Part B

Now we are continuously adding $1000 per year, yielding a differential equation of:

$$\frac{dM(t)}{dt} = .04M(t) + 1000$$

Problem 2

So we have that $P(t)$ is the number of infected people as a function of time and N is the total population, which is fixed. We are also given that rate at which people are infected (which is $\frac{dP(t)}{dt}$) is given by the product of the number of people who have been infected, which is just $P(t)$ and the number of people who have not been infected, which is just $N - P(t)$, the total number of people minus the number infected. Thus our differential equation is just:

$$\frac{dP(t)}{dt} = P(t)\,(N - P(t))$$

Problem 4

We are given that $G(t)$ is the number of tons of garbage in the dump. We also add T tons per month, and the rate at which garbage is being hauled off is proportional to the

tonnage at the site, so it will be of the form $\alpha G(t)$ where α is some position constant. We will let our variable t be the number of months, rather than the number of years. Then our differential equation is:

$$\frac{dG(t)}{dt} = T - \alpha G(t)$$