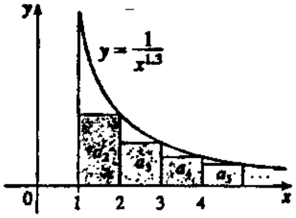
Problem Set #16

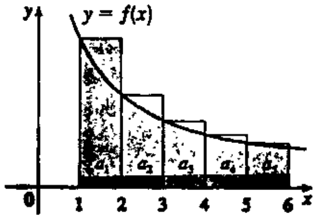
8.3: 1, 2, 5, 15, 16, 18, 19, 20, 22 + Series Handout #9

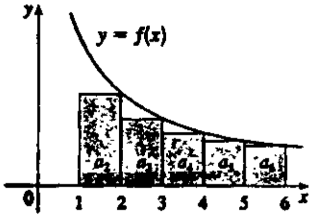
1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (5.10.2) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





5. $\displaystyle\sum_{n=1}^{\infty} n^b$ is a p-series with $p = -b$. $\displaystyle\sum_{n=1}^{\infty} b^n$ is a geometric series. By (1), the p-series is convergent if $p > 1$. In this case, $\displaystyle\sum_{n=1}^{\infty} n^b = \sum_{n=1}^{\infty} (1/n^{-b})$, so $-b > 1 \iff b < -1$ are the values for which the series converge. A geometric series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$, so $\displaystyle\sum_{n=1}^{\infty} b^n$ converges if $|b| < 1 \iff -1 < b < 1$.

15. $f(x) = \dfrac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1 + \ln x}{x^2 (\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series diverges.

16. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3+4}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

18. $\dfrac{\sin^2 n}{n\sqrt{n}} \leq \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ converges by the Comparison Test.

19. $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

20. $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

22. Let $a_n = \dfrac{1}{n^3 - n}$ and $b_n = \dfrac{1}{n^3}$. Then $\displaystyle\sum_{n=2}^{\infty} a_n$ and $\displaystyle\sum_{n=2}^{\infty} b_n$ are series with positive terms and

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3 - n} = 1 > 0$. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$ is a convergent p-series without the $n = 1$ term ($p = 3 > 1$),

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$ is convergent by the Limit Comparison Test.

9) a) Note that $\frac{1}{n+2} > \frac{1}{n+n}$ for $n>2$ so $\frac{1}{n+2} > \frac{1}{2n}$ for $n>2$

so $\sum_{n=3}^{\infty} \frac{1}{n+2} > \sum_{n=3}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{n}$

But $\sum \frac{1}{n}$ diverges because it is the harmonic series and $\sum \frac{1}{n+2}$ has infinitely many terms greater than the harmonic series so it diverges as well.

b) Note that $\frac{1}{\sqrt{n^2+10}} > \frac{1}{\sqrt{n^2+n^2}}$ for $n>3$

But $\frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}$ so again

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+10}} > \sum_{n=3}^{\infty}\frac{1}{n\sqrt{2}} = \frac{1}{\sqrt{2}}\sum_{n=3}^{\infty}\frac{1}{n}$ which diverges

so $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+10}}$ must also diverge by the same logic as above.

c) $\sum_{n=}^{\infty}\frac{1}{n\ln n} = \quad + \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots \frac{1}{n\ln n}$

note that $\frac{1}{n\ln n}$ is strictly decreasing

Now we want to group the terms in a very specific way. Letting $a_n = \frac{1}{n\ln n}$ then

$$\sum_{n=2}^{\infty} a_n = 0 + a_2 + (a_3+a_4) + (a_5+\cdots+a_8) + (a_9+\cdots+a_{16}) + \cdots$$

$$> 0 + a_2 + 2a_4 + 4a_8 + 8a_{16} + \cdots + 2^k a_{2^{k+1}}$$

$$> \frac{1}{2}(0 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots + 2^k a_{2^k})$$

$$> \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{2^k \ln 2^k}\cdot 2^k\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\ln 2^k} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k\cdot\ln 2}$$

$$> \frac{1}{2\ln 2}\sum_{k=1}^{\infty}\frac{1}{k}$$ $\leftarrow$ but this is the harmonic series and diverges. Therefore $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ also diverges.

c) (continued) letting $a_n = \frac{1}{n(\ln n)^p}$

by the same logic as before we group the things.

In fact for a monotonically decreasing series $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$ the series $\sum_{n=1}^{\infty} a_n$ converges iff the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots 2^k a_{2^k}$$

converges.

We showed previously that if it diverges so does $\sum_{n=1}^{\infty} a_n$. The proof of the other direction uses the comparison test showing that $\sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$ so if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges so does $\sum_{n=1}^{\infty} a_n$.

We can apply this to $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ which converges if

$$\sum_{k=0}^{\infty} \frac{2^k}{2^k (\ln 2^k)^p} = \sum_{k=0}^{\infty} \frac{2^k}{2^k (k \ln 2)^p} = \sum_{k=0}^{\infty} \frac{1}{(\ln 2)^p k^p} = \frac{1}{(\ln 2)^p} \sum_{k=0}^{\infty} \frac{1}{k^p}$$

which converges for $p > 1$.

Extra credit:

apply what we used above. Then $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ converges if $\sum_{k=0}^{\infty} \frac{2^k}{2^k \ln 2^k (\ln(\ln 2^k))}$ converges

$$= \sum_{k=0}^{\infty} \frac{1}{k \ln 2 (\ln (k \ln 2))} \frac{1}{\ln 2} \sum \frac{1}{k(\ln k + \ln \ln 2)}$$

which looks similar to $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ and diverges.

To see this we can apply it again replacing the ks with 2^z and multiply the general term by 2^z. That yields

$$\frac{1}{\ln 2} \sum \frac{2^z}{2^z(\ln 2^z + \ln \ln 2)} = \frac{1}{(\ln 2)^2} \sum_{z=0}^{\infty} \frac{1}{z + \frac{\ln \ln 2}{\ln 2}}$$

But this diverges just as the harmonic series diverges, apply integral test.