**11.9, 16**

The Taylor series for $f(x) = \sqrt[3]{x}$ about $x = 27$ is

$$f(x) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2!}(x-27)^2 + \cdots.$$

Since $f(27) = 3$, $f'(27) = \frac{1}{3}27^{-\frac{2}{3}}$, and $f''(27) = -\frac{2}{9}27^{-\frac{5}{3}}$, we see that

$$f(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2157}(x-27)^2 + \cdots.$$

Based on this, we try approximating $f(28)$ by the first degree Taylor polynomial of $f(x)$ around $x = 27$. The error is then bounded by

$$|R_1(28)| \leq \frac{M}{(1+1)!}|28-27|^2 = \frac{M}{2}.$$

Now $M$ is an upper bound for $|f''(x)| = \frac{2}{9}x^{-\frac{5}{3}}$ on the interval $x \in [27, 28]$. The maximum value of this function occurs at the left endpoint (since the function is decreasing), and therefore

$$M \leq \frac{2}{9}27^{-\frac{5}{3}} = \frac{2}{2187}.$$

Plugging this into the error formula, we see that

$$|R_1(28)| \leq \frac{1}{2187} < \frac{1}{2000} = 0.0005,$$

which means that the first degree Taylor polynomial gives us our answer to within three decimal places. Thus,

$$\sqrt[3]{28} = 1 + \frac{1}{27} = 1.037\ldots,$$

to three decimal places.

**11.9, 18a**

The error from the Taylor remainder formula is bounded by

$$|R_2(x)| \leq \frac{M}{(2+1)!}|x-0|^{2+1},$$

where $M$ is an upper bound for $f^{(n+1)}(x) = e^x$ on the interval $x \in [0, b]$. To bound this above, use the bound $e < 3$. Thus the error is bounded by

$$|R_2(x)| \leq \frac{3^x}{(2+1)!}|x|^{2+1},$$

for $x \in [0, b]$. If you let $b = \frac{1}{10}$, then this error will be bounded above by

$$|R_2(x)| \leq \frac{3^{\frac{1}{10}}}{3!}\frac{1}{10^3},$$

and since $3^{\frac{1}{10}} < 3$, we have that

$$|R_2(x)| \leq 0.0005,$$

for $x \in \left[0, \frac{1}{10}\right]$. This is the three decimal place accuracy that we want, and so $b = \frac{1}{10}$ will work. (Note that other possible values of $b$ would work as well.)

**11.9 20a**

The function x is the first degree Taylor polynomial for $ln(1+x)$ about $x = 0$. The error is therefore bounded by Taylor's remainder term formula with $n = 1$, which says that it is less than

$$\frac{M}{2!}|x|^2.$$

Here $M$ is an upper bound for $|f''| = \frac{1}{(1+x)^2}$ on the interval in question, which you can check is bounded above by 1 throughout this interval. Thus an upper bound is $|x|^2/2$, which, for $|x| < 0.01$, is bounded by $0.0001/2 = 0.00005$, and that is the maximum error.

**11.9 22**

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k.$$

**A**

Using the binomial series formula,

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots.$$

**B**

Substituting $-x^2$ for $x$ in **A**, we see that

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots.$$

Integrating this series term by term, we see that

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^4 + \cdots.$$

We see that the constant of integration $C$ must have been equal to zero (by looking at the value of both sides of the equation at $x = 0$, for example, since $\sin^{-1} 0 = 0$, and since the series, without the constant, vanishes at $x = 0$).