Series Review

Math 1b

January 11, 2006

1 Tests for Convergence

1.1 Strategies

Take all lists with a grain of salt.

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric sereis, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation by be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or geometric series, then one of the **comparison tests** should be considered. In particular, if a_n is a **rational function or algebraic function** of n (involving roots of polynomials), then series should be compared with a p-series. The value of p should be chosen by keeping only the highest powers of the numerator and denominator. The comparison tests apply only to series with **positive terms**, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for **absolute convergence**.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the **Test for Divergence** should be used.
5. If the series is of the form $\sum (-1)^n b_n$ or $\sum (-1)^{n-1} b_n$, then the **Alternating Series Test** is an obvious possibility.
6. Series that involve **factorials or other products** (including a constant raised to the nth power) are often conveniently tested using the **Ratio Rest**. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should *not* be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the **Integral Test** is effective (assuming that f is positive and decreasing).

Determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}.$

2. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n+1}{3n}.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{9^n}$

Note. Test of Divergence cannot be used to show convergence. Likewise, the Alternating Series Test cannot be used to show divergence.

1.2 Advanced Strategies

Always remember that

- $|\sin x| \leq 1$ for all x.
- $|\cos x| \leq 1$ for all x.
- For any $p > 0$, $\ln x \leq x^p$ for large x.
- For any $p > 0$, $x^p \leq e^x$ for large x.

These inequalities help us find appropriate series to compare to.

4. Test the series for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2 + 4n}$.

5. Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$.

2 Power Series

2.1 Recognizing a series as the evaluation of a power series

Suppose the series $\sum_{n=1}^{\infty} (x+1)^n c_n$ converges at $x = -4$ and diverges at $x = 3$.

6. What is the center of the series?

7. What can be said about the radius of convergence of the power series $\sum_{n=1}^{\infty} (x+3)^n c_n$?

8. Determine whether or not the following series converge.

(i) $\sum_{n=1}^{\infty} (-2)^n c_n$

(ii) $\sum_{n=1}^{\infty} c_n$

(iii) $\sum_{n=1}^{\infty} 4^n c_n$

(iv) $\sum_{n=1}^{\infty} (-4)^n c_n$

2.2 Representing functions as power series

Famous Series

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.
- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x.
- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x.
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

9. Find the Taylor series of e^{-x^2} centered at $x = 0$.

3 Estimation Problems

The main tool we can use to estimate the error of using partial sums to approximate the limit of a series is the following:

Theorem. **Alternating Series Estimation Theorem**

If $s = \displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ is the sum of an alternating series (all $b_n > 0$) that satisfies

1. $b_{n+1} \leq b_n$, and
2. $\lim_{n\to 0} b_n = 0$,

then

$$|s - s_n| \leq b_{n+1}$$

10. Approximate the value of $e^{-0.1}$ with $|\text{error}| < 0.01$.