Math 1b – Sequences and series summary

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1 Sequences

(Stewart p. 557)

Notations for a sequence:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

or

$$\{a_n\}_{n=1}^{\infty}.$$

The numbers a_n are called the **terms** of the sequence.

1.1 Limit of a sequence

(Stewart p. 559)

A sequence $\{a_n\}_{n=1}^{\infty}$ has the **limit** L if we can make the terms a_n as close as we like to L by choosing n to be sufficiently large. We write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty,$$

and we say that the sequence **converges to** L, or is **convergent**. If the sequence does not converge, we say that it **diverges**, or is **divergent**.

If the terms a_n grow arbitrarily large as n increases, then we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty \text{ as } n \to \infty,$$

and we say that the sequence **diverges to infinity**. Similarly for $-\infty$.

1.1.1 Connection to limits involving real numbers

(Stewart p. 559)

If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for integers n, then $\lim_{n\to\infty} a_n = L$. For example, we know that

$$\lim_{x\to\infty} \frac{x}{x^3+4} = 0,$$

so we can just "replace every x by n" and conclude that

$$\lim_{n\to\infty} \frac{n}{n^3+4} = 0.$$

1.2 Operations on sequences

(Stewart p. 560)

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences, then we can add, subtract, and multiply them and get the expected answers. For example

$$\lim_{n\to\infty}\left(\frac{n^2+n}{2n^2+3}-2^{1/n}\right)=\lim_{n\to\infty}\left(\frac{n^2+n}{2n^2+3}\right)-\lim_{n\to\infty}2^{1/n}.$$

Division works too, as long as we don't divide by zero on the right hand side.

1.3 Squeeze theorem

(Stewart p. 560)

The squeeze theorem is much the same as for functions of real numbers. If

$$a_n \le b_n \le c_n$$

for $n \ge n_0$, and if

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then the values of b_n are 'squeezed' between a_n and c_n, so we conclude that

$$\lim_{n\to\infty} b_n = L$$

1.4 Monotone convergence theorem

(Stewart p. 562)

A sequence $\{a_n\}_{n=1}^{\infty}$ is called **monotonic** if it is either **increasing**, i.e.

$$a_1 < a_2 < a_3 < \cdots,$$

or if it is **decreasing**, i.e.

$$a_1 > a_2 > a_3 > \cdots.$$

That is, monotonic means 'doesn't change direction'.

A sequence $\{a_n\}_{n=1}^{\infty}$ is called **bounded** if there is a constant R so that

$$-R \le a_n \le R$$

for all n. That is, the sequence stays between two finite bounds.

The **monotone convergence theorem** states that if a sequence is *monotonic* and *bounded*, then it is *convergent*. The intuition is that the numbers must be getting squished closer and closer together – they have no choice but to converge.

2 Series

2.1 Partial sums and the definition of a series

(Stewart p. 567)

Given a sequence $\{a_n\}_{n=1}^{\infty}$, we may form the **partial sums**:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ &\vdots \\ s_N &= a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n. \end{aligned}$$

If the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges to a limit L, then we say that the **series** $\sum_{n=1}^{\infty} a_n$ **converges**, and that its sum is L. Another way to say this is that we are defining

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n,$$

if the limit on the right hand side exists. (Note the similarity to the definition of an improper integral.)

2.2 The geometric series

(Stewart p. 569)

The **geometric series** is the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .$$

It is divergent for $|r| \geq 1$ and convergent for $|r| < 1$, in which case the sum is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

2.3 A simple divergence test

(Stewart p. 572)

If the limit

$$\lim_{n \to \infty} a_n$$

is nonzero or does not exist at all, then the series $\sum_{n=0}^{\infty} a_n$ cannot possibly converge. For example, the following series is divergent:

$$\sum_{n=0}^{\infty} \left(1 - \frac{1}{n}\right).$$

2.4 Operations on series

(Stewart p. 573)

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then we can add and subtract them term-by-term and get the expected answers. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

2.5 Convergence tests for series with *positive* terms

2.5.1 The integral test

(Stewart p. 578)

If $f(x)$ is a *continuous*, *positive*, *decreasing* function on $[1, \infty)$, and if $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \qquad \text{if and only if} \qquad \int_1^{\infty} f(x)\,dx \text{ converges.}$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \qquad \text{because} \qquad \int_1^{\infty} \frac{1}{x}\,dx \text{ diverges.}$$

2.5.2 The p-test

(Stewart p. 579)

Just as in the case of improper integrals,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent for } p > 1, \\ \text{divergent for } p \leq 1. \end{cases}$$

2.5.3 The comparison test

(Stewart p. 580)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with *positive* terms, and that $a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.
- If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.

In other words, if you start with a convergent series and make its terms smaller, it stays convergent; and if you start with a divergent series and make its terms bigger, it stays divergent.

Hint: If your inequality ends up pointing the wrong way, don't just stare at it and hope it changes direction — try again with a different test!

2.5.4 The limit comparison test

(Stewart p. 582)

The limit comparison test is a turbo-charged version of the comparison test. If the comparison test doesn't quite work, but seems to be close, consider using the limit comparison test.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with *positive* terms, and if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where $0 < c < \infty$ (i.e. the sequence of ratios has a finite, nonzero limit), *then* either both series converge or both series diverge.

For example,

$$\lim_{n\to\infty} \frac{(n^2+5)/(n^4+5)}{1/n^2} = 1,$$

and we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so we conclude that also

$$\sum_{n=1}^{\infty} \frac{n^2+5}{n^4+5}$$

converges.

2.6 Alternating series

(Stewart p. 586)

An **alternating series** is a series whose terms are alternately positive and negative. For example,

$$1 - 47 + 0.34 - \frac{1}{\pi} + 2 - \cdots .$$

2.6.1 The alternating series test for convergence

(Stewart p. 587)

If $\{b_n\}_{n=1}^{\infty}$ is a sequence such that

- $b_1 \geq b_2 \geq b_3 \geq \cdots$, and
- $\lim_{n\to\infty} b_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

2.6.2 The alternating series remainder estimate

(Stewart p. 588)

Suppose that $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is an alternating series satisfying the above two conditions for the alternating series test, and let $\{s_N\}_{N=1}^{\infty}$ be the sequence of partial sums. Then

$$|s - s_N| \leq b_{N+1};$$

in other words, the error committed by only taking the first N terms in the series (instead of *all* of them) is at worst the next term in the sequence (b_{N+1}).

2.7 Absolute convergence

(Stewart p. 589)

The series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if the corresponding series with positive terms,

$$\sum_{n=1}^{\infty} |a_n|,$$

is convergent.

This concept is useful for two reasons:

- If a series is absolutely convergent, then it is actually *convergent. (Stewart p. 590)*
- We have many more tools for dealing with series whose terms are all positive.

2.7.1 The ratio test

(Stewart p. 591)

Suppose that $\sum_{n=1}^{\infty} a_n$ is a series (terms are not necessarily positive). Consider the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

- If the limit exists and is < 1, then the series is absolutely convergent (and hence convergent).
- If the limit exists and is > 1 (including the case where it diverges to infinity), then the series is divergent.
- If the limit $= 1$, then the ratio test tells us precisely *nothing.*

3 Power series

(Stewart p. 594)

A **power series** in $(x-a)$ (or a power series **centered at** a) is a series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots . \end{aligned}$$

The numbers $\{c_n\}_{n=1}^{\infty}$ are called the **coefficients** of the power series.

Every power series has a **radius of convergence**, often written R. The power series converges for all x inside the open interval $(a-R, a+R)$, and it diverges for $x < a-R$ and $x > a+R$. It may or may not converge at the endpoints $a \pm R$. The totality of x for which the series converges is called the **interval of convergence**.

The radius of convergence may also be zero, in which case the series converges only for $x = a$; or it could be infinite (we write $R = \infty$), in which case the series converges for all x.

The ratio test is often useful in determining the radius of convergence.

3.1 Differentiating and integrating power series

(Stewart p. 601)

If the power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergence R, then we may differentiate and integrate it term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}, \\ \int f(x)\,dx &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C. \end{aligned}$$

These two series both have the same radius of convergence R as the original series. *Warning:* their *interval* of convergence may differ from that of f.

For example, starting with the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, by differentiating we obtain

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots ,$$

and by integrating we obtain

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots .$$

All three series have radius of convergence 1, but their intervals of convergence are respectively $(-1,1)$, $(-1,1)$ and $[-1,1)$.

3.2 Taylor and Maclaurin series

(Stewart p. 605)

If $f(x)$ is a function whose derivatives $f^{(n)}(x)$ exist at $x = a$ for all $n \geq 1$, then the **Taylor series** of $f(x)$ centered at $x = a$ is the power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

where the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Warning: The Taylor series *may or may not* converge for $x \neq 0$. Even if it does converge, it might not converge to the function $f(x)$ that we started with. Nevertheless, for most of the examples we deal with in this course, the Taylor series for $f(x)$ *does* converge to $f(x)$ in some interval.

A **Maclaurin series** is just a Taylor series with $a = 0$.

3.2.1 Some Taylor series you should know

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}$$

3.3 The binomial series

(Stewart p. 617)

The **binomial series** is a Taylor series for the function $f(x) = (1+x)^k$, where k is any real number:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n,$$

where $\binom{k}{n}$ are the **binomial coefficients**,

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}.$$

The radius of converge of this series is 1, except when k is a positive integer, in which case the radius is infinite.