Review Problems for the First Exam

Math 1b

October 20, 2005

1 Improper Integrals

1. Two of your young classmates are having some trouble with improper integrals. They are discussing $\int_0^\infty f(x)\,dx$ where f is defined, positive, and bounded on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$.

   Margaret believes that $\int_0^\infty f(x)\,dx$ ought to diverge. She reasons that if f is positive then then the accumulated area keeps increasing, even if only by a little bit, so we can't get anything other than infinity since $\int_0^b f(x)\,dx$ increases with b and we need to let b go to infinity.

   Amani, on the other hand, is convinced that if $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ ought to converge. After all, he reasons, the rate at which area is accumulating is going to zero. That should be enough assure convergence.

   Margaret and Amani ask you for assistance. Explain very clearly the errors each of them are making.

   There is really not enough information given about $\int_0^\infty f(x)\,dx$ to draw any conclusion. Illustrate this by providing two integrals of this form (satisfying the conditions given) one of which converges and one of which diverges.

2. For what values of p does $\int_1^\infty \frac{dx}{x^p}$ converge?

3. For what values of p does $\int_0^1 \frac{dx}{x^p}$ converge?

4. For what values of p does $\int_0^\infty \frac{dx}{x^p}$ converge?

5. Let A be the region bounded above by the $x-$axis, below by the curve $y = \frac{1}{x}$ and on the right by the line $x = -1$.

   (a) Does the region A have finite area? If so, what is it?

   (b) Let V be the volume generated by revolving area A about the $x-$axis. (V is sometimes called *Gabriel's Horn.*) Find the volume of V. Curiously, it is finite, even though A is infinite ...

(c) Let W be the volume generated by revolving area A about the $y-$axis. Is the volume finite? If so, what is it?

6. Does the integral $\int_0^{10} \frac{dx}{x^2-3}$ converge?

7. Does the integral $\int_0^{\infty} e^{-x^3} dx$ converge?

2 Slicing

Note: for any of these problems you ought to be able to write a Riemann Sum approximating the quantity sought and, by taking the limit of the Riemann Sum, be able to obtain the integral giving the sought after quantity.

8. A beam of light is shining onto a screen creating a disk of radius 50 cm. The intensity of light is greatest at the center an diminishes away from the center. It the intensity of light at a distance r from the center of the beam is given by $f(r)$ watts./square cm, find an expression for the total wattage of the beam's image on the screen.

9. At Three Aces pizzeria the chef tosses lots of garlic on the pizza. The density of garlic varies with x, the distance from the center of the pizza, and is given by $g(x) = \frac{x}{(x^3+2)^2}$ ounces per square inch of pizza. If the pizza is 14 inches in diameter, and Three Aces cuts six slices from each pizza, how much garlic is on once slice of pizza?

10. A very thin, lighted pole 10 feet tall is placed upright in a family's backyard to attract insects. At one moment, the density of these insects is given by $\rho(r) = \frac{1.3}{\pi(r+1)}$ insects per cubic foot where r measures the number of feet from the pole. (This distance is measured to the closest distance on the pole.)

    (a) How many insects are within 5 feet of the pole at a height of 10 feet or less?

    (b) How many insects are within 5 feet of the pole at a height of 10 feet or more?

3 Volumes

11. Let A be the region bounded above by the $y = \tan x$, below by the $x-$axis, and on the right by $x = \pi/4$.

    Find an integral giving the volume obtained when the region A is rotated about

(a) the $y-$ axis
(b) the line $x = \pi$
(c) the line $y = -3$

You need not evaluate the integrals.

12. Consider the region R in the plane bounded by the parabola $x = y^2$ and the line $x = 9$. Now consider an object in three-dimensional space with R as its base. Its cross-sections perpendicular to R are semicircles with diameter on the base region. What is the volume of the object?

13. Derive the formula $V = \frac{4}{3}\pi R^3$ for the volume of a sphere using the method of slicing.

14. Find the volume of the cap of a sphere with radius r and height h. (See the picture on p. 464 of Stewart: #23.)

15. A conical frustum is the region formed by starting with a cone and then chopping it off by a plane parallel to the base. Find a formula for the volume of a conical frustum whose radius at the top is r, whose radius at the bottom is R, and whose height (the distance between the two bases) is h.

16. Find the volume of the volume obtained by revolving the plane region enclosed by $x = y^2$ and $x = 2y + 3$ around the x-axis.

17. Use the method of washers to find the region obtained by taking the plane region enclosed by $y = x$, $y = \sqrt{x}$ about the line $y = 1$. When you finish, find the same problem using the method of shells, and check that your answers agree.

18. A Wisconsin cheese factory makes its cheese in solid cylinders of radius 2 inches. A wedge of cheese is cut from the cylinder by chopping through the diameter of the base at an angle of 45 degrees with the base. Find the volume of the wedge of cheese.

    (Hint: the base of the wedge is a semicircle of radius 2. The cross sections perpendicular to the base are isosceles right triangles.)

4 Arc Length

19. Find the length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$.

20. Give a definite integral representing the length of the parametric curve $x = t^3$, $y = t^4$, $0 \leq t \leq 1$.

21. Find the distance traveled by a particle with position $(x, y) = (\cos^2 t, \cos t)$ as t varies in the time interval $[0, 4\pi]$. Compare this answer with the length of the curve.

5 Work problems

22. Filled with the desire to be alone, you have climbed a tree which is 10 meters high. So that you will be able to stay there for a while, you have arranged a basket containing lunch and some reading materials, weighing 15 kg in all. Before climbing up the tree you have attached a sturdy chain to the basket – the chain weighs 1 kg per meter. How much work do you do pulling the picnic basket up to you via the chain?

23. Consider a spherical tank with radius 10 meters which is completely filled with water. Calculate the work done in pumping out the water through the top of the tank. (The mass density of water is 1000 kilograms per cubic meter, and the acceleration due to gravity is 9.8 meters per second squared.)

24. Amelia and Beulah are city dwellers who have set up pulley systems to get their groceries delivered without walking the stairs. Amelia pulls her basket filled with 12 pounds of groceries up to her 40 foot high balcony Beulah pulls her basket filled with 16 pounds of cleaning supplies up to her 30 foot high window. Assuming both women use ropes weighing 0.2 lbs/ft, whose task requires more work? How much more work? (Assume friction is negligible.)

25. As part of the pasteurization and homogenization process, milk is stored in a large tank. The top of the tank is a 6 foot by 5 foot rectangle and the cross sections of the tank are semicircles of radius 3. (Picture a tank that looks like a hollowed out log 5 ft. long and 3 feet in diameter chopped in half the long way and lying on the ground on its side. See the supplement p. 876 for a picture.)

    The milk is to be pumped out of this holding tank. If the tank is filled to the brim, how much work is required to empty it? The weight-density of milk is 64.5 lb/ft^3.

26. A fountain in a Newport mansion has a shape given by revolving region bounded by the $x-$ and $y-$ axis, $y = 2$ and the curve $y = \ln x$ about the $y-$axis. The fountain is 1 foot tall; the radius of its base is 1 foot. The fountain sits on the front lawn. After a hurricane it contains murky silty water.

    The weight of the silt-laden water varies with the height and is given by $\rho(y)$ lbs/ft^3 where y is the distance from the lawn.

    If the fountain is filled with murky water to a height of 1.8 feet,

    (a) what is the weight of the stuff in the fountain?
    (Your answer should be in terms of an integral involving $\rho(y)$.)

    (b) how much work is required to empty the fountain by emptying the murky water over the top rim of the fountain?

(Your answer should be in terms of an integral involving $\rho(y)$.)