Name: SOLUTIONS

Final Exam for Mathematics 1b

May 26, 2005

Problem	Points	Score
1	7	
2	13	
3	14	
4	10	
5	9	
6	6	
7	7	
8	8	
9	10	
10	16	
Total	100	

- You have three hours for this exam.
  Work carefully and efficiently. Do not spend an inordinate amount of time on any one problem.
- Please show all your work on this exam paper.
  Unless instructed otherwise, you must clearly indicate your line of reasoning in order to get full credit.
- If you have work on the back of a page, indicate that on the exam cover.
- NO calculators are permitted.
- Think clearly and do well!

Please circle your section.

MWF 10	Andrew Lobb	MWF 10	Rene Reinbacher	MWF 11	Matthew Leingang
TTH 10	Robin Gottlieb	TTH 11:30	Andreea Nicoara	TTH 11:30	Robin Gottlieb

Final Examination

May 26, 2005

1. (8 points)

The function f is defined by

$$f(x) = \frac{e^x + e^{-x}}{2}$$

(a) Write down the Taylor series for $f(x)$ centered at $x = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(1 + (-1)^n \right)$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} \cdot 2$$

$$= \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$$

(b) For what values of x does the series in (a) converge?

$$\lim_{m\to\infty} \left| \frac{\frac{x^{2(m+1)}}{[2(m+1)]!}}{\frac{x^{2m}}{(2m)!}} \right| = \lim_{m\to\infty} \left| \frac{x^2}{(2m+1)(2m+2)} \right| = 0 \quad ,$$

the series converges for $x \in (-\infty, +\infty)$

(c) There are two ways to do part (a). One is to use series that you know and manipulate them; the other is to find the Taylor series by taking derivatives. Will these two methods give you the same answer?

Yes,

(d) Find $f^{(n)}(0)$ for $n = 2, 13$, and 14.

$$\frac{f^{(2m)}(0)}{(2m)!} = \frac{1}{(2m)!} \Rightarrow f^{(2m)}(0) = 1 . \quad f^{(2m+1)}(0) = 0 ,$$

$f^{(2)}(0) =$ 1 $\quad f^{(13)}(0) =$ 0 $\quad f^{(14)}(0) =$ 1

(e) Let $i = \sqrt{-1}$. What familiar function is $f(ix)$? (Justify your answer).

$$e^{ix} = \cos x + i \sin x .$$

$$f(ix) = \frac{(\cos x + i \sin x) + (\cos(-x) + i \sin(-x))}{2}$$

$$= \frac{2\cos x + i \sin x - i \sin x}{2} = \cos x .$$

2. Differential equations medley (13 points)

(a) (4 points) Consider the differential equation

$$y' = 2(y - x) + \frac{y}{x}.$$

Is $y = xe^{2x} + x$ a solution to this differential equation? Explain by showing your work.

$y = xe^{2x} + x$

$L.H.S = y' = e^{2x} + x \cdot e^{2x} \cdot 2 + 1 = e^{2x}(1+2x) + 1$

$R.H.S = 2(xe^{2x} + x - x) + \frac{xe^{2x} + x}{x} = 2xe^{2x} + e^{2x} + 1$

$= e^{2x}(1+2x) + 1$

$\Rightarrow L.H.S = R.H.S$

$\Rightarrow y = xe^{2x} + x$ is a solution.

(b) (4 points) Due to a malfunctioning mechanism in an industrial plant upriver from a lake, the lake has been contaminated with pollutants. At a time we'll designate as $t = 0$ there are 8 x 10^5 grams of pollutant in the lake. The problem is now being addressed, and the concentration of pollutants in the incoming water is decreasing. The concentration of pollutants in the incoming water at time t is given by $\frac{1}{100t^2+1}$ grams per liter. The 9 x 10^7 liter lake is fed at a rate of 50,000 liters per day, and the well-mixed water leaves at the same rate. Let $P(t)$ be the number of grams of pollutant in the lake at time t. Write a differential equation involving $\frac{dP}{dt}$ and reflecting the situation. You need not solve.

$$\frac{dP}{dt} = 50000 \cdot \frac{1}{100t^2+1} - \frac{P}{9 \times 10^7} \cdot 50000$$

$P(0) = 8 \times 10^5$ gram.

Differential equation:________________ Initial condition:__________

(c) (5 points) Solve:

$$\frac{dy}{dx} = \frac{e^{x+y^2}}{y\sqrt{3+e^x}}$$

(You need not check your answer.)

$$\frac{dy}{dx} = \frac{e^{x+y^2}}{y\sqrt{3+e^x}} = \frac{e^x}{\sqrt{3+e^x}} \cdot \frac{e^{y^2}}{y}$$

$$\Rightarrow \int \frac{y\,dy}{e^{y^2}} = \int \frac{e^x}{\sqrt{3+e^x}}\,dx \quad \text{- - - } (*)$$

L.H.S of $(*)$

$$= \frac{1}{2}\int \frac{d(y^2)}{e^{y^2}} = -\frac{1}{2}e^{-y^2} + C_1$$

R.H.S of $(*)$

$$= \int \frac{d(e^x)}{\sqrt{3+e^x}} = \frac{1}{-\frac{1}{2}+1}\left(3+e^x\right)^{-\frac{1}{2}+1} = 2\sqrt{3+e^x} + C_2$$

$(*) \Longleftrightarrow$

$$C - \frac{1}{2}e^{-y^2} = 2\sqrt{3+e^x}$$

$$\Longleftrightarrow \quad e^{-y^2} = 2\left(C - 2\sqrt{3+e^x}\right)$$

3. Short Answer Questions (14 points) : You do not need to show your work.

A. (8 points) To each differential equation below, match the picture that best corresponds to its slope field.

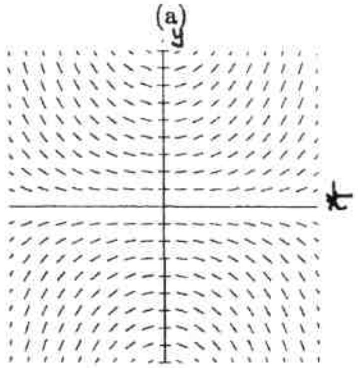
(a) $\frac{dy}{dt} = y^2 - 1$ ___C___

(b) $\frac{dy}{dt} = t^2 - 1$ ___e___

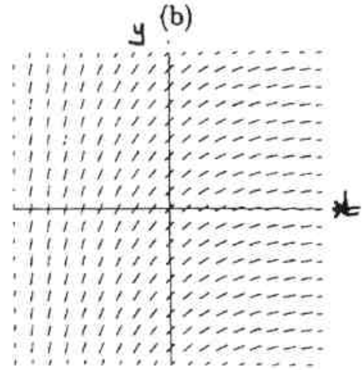
(c) $\frac{dy}{dt} = t^2 + 1$ ___d___ $\frac{dy}{dt} = t^2 + 1$

(d) $\frac{dy}{dt} = ty$ ___a___

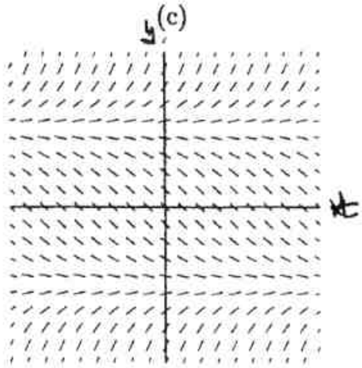
(a)



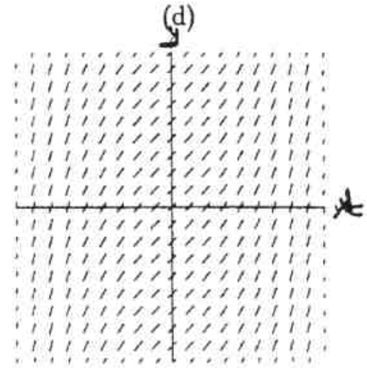
(b)



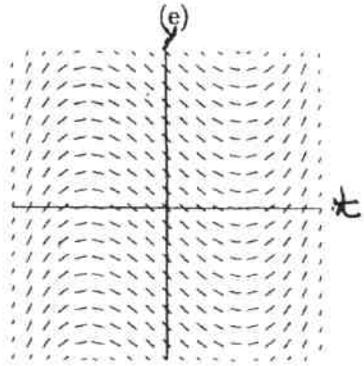
(c)



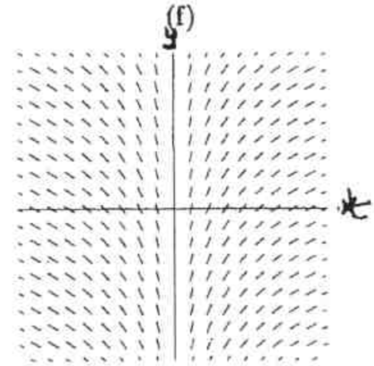
(d)



(e)



(f)



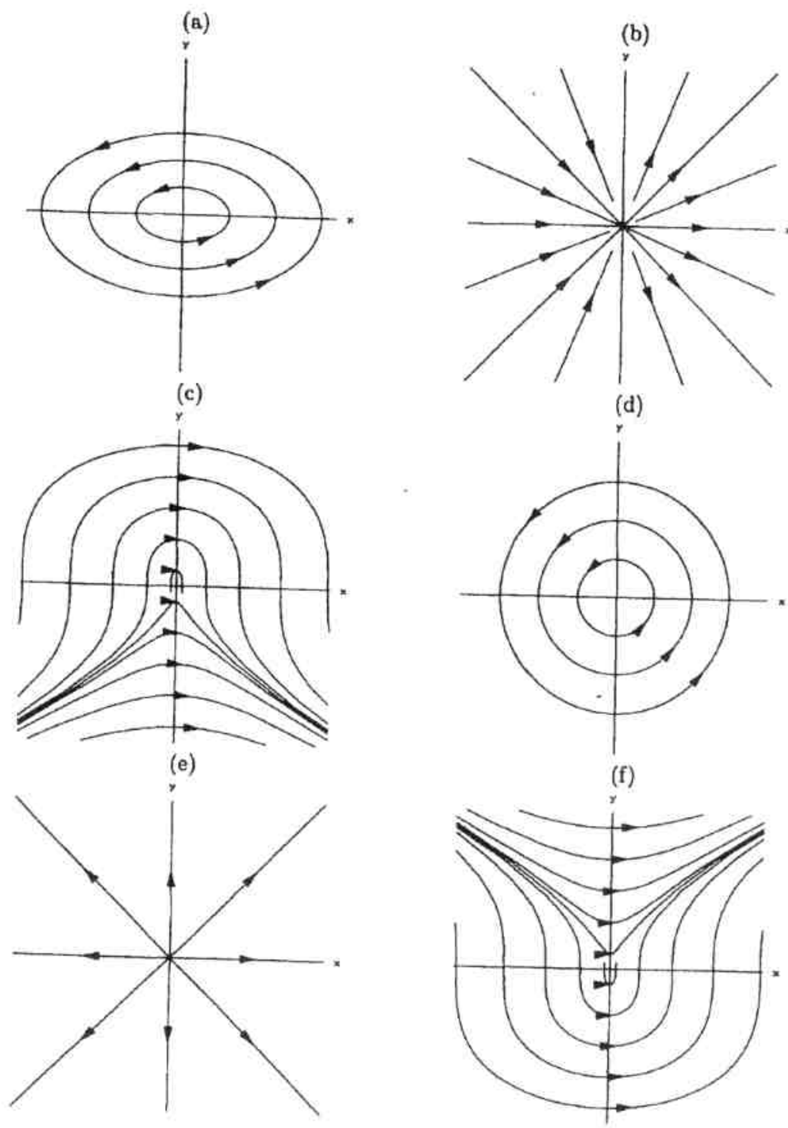
B. (6 points) Below are two systems of differential equations. To each system match the appropriate picture of solution trajectories in the xy−plane.

$$I.\quad \frac{dx}{dt} = y^2 \qquad \frac{dy}{dt} = -x$$

$$II.\quad \frac{dx}{dt} = x^2 \qquad \frac{dy}{dt} = xy$$

Graph __c__ corresponds to system I.

Graph __b__ corresponds to system II.



4. (10 points)

A. Which of these is the best method for approximating $\sqrt{7}$ by hand?

(a) Expand the series for $f(x) = \sqrt{x}$ at $a = 0$ and evaluate at $x = 7$.
(b) Expand the series for $f(x) = \sqrt{x}$ at $a = 1$ and evaluate at $x = 6$.
(c) Expand the series for $f(x) = \sqrt{x}$ at $a = 4$ and evaluate at $x = 3$.
(d) Expand the series for $f(x) = \sqrt{x}$ at $a = 4$ and evaluate at $x = 7$.
(e) Expand the series for $f(x) = \sqrt{x}$ at $a = 7$ and evaluate at $x = 7$.
(f) Expand the series for $f(x) = \sqrt{x}$ at $a = 9$ and evaluate at $x = -2$.
(g) Expand the series for $f(x) = \sqrt{x}$ at $a = 9$ and evaluate at $x = 7$.

(g)

$$\sqrt{a+(x-a)} = \sqrt{a}\sqrt{1+\frac{x-a}{a}} = \sqrt{a}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\cdot\left(\frac{x-a}{a}\right)^n$$
$$= \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n} a^{\frac{1}{2}-n}(x-a)^n.$$

B. Follow through on your answer to part A. Find the second degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at the appropriate value of a.

$a = 9$, $x = 7$

$$T_2 = \sqrt{9}\left(\binom{\frac{1}{2}}{0}\left(\frac{x-9}{9}\right)^0 + \binom{\frac{1}{2}}{1}\cdot\left(\frac{x-9}{9}\right) + \binom{\frac{1}{2}}{2}\left(\frac{x-9}{9}\right)^2\right)$$
$$= 3\cdot\left(1 + \frac{1}{2}\cdot\frac{(-2)}{9} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{-2}{9}\right)^2\right)$$
$$= 3\left(1 - \frac{1}{9} - \frac{1}{8}\cdot\frac{4}{81}\right) = 3\left(\frac{143}{2\cdot 9^2}\right) = \frac{143}{54}$$

Second degree Taylor polynomial approximation of $\sqrt{x}$: $3\left(1+\frac{1}{2}\cdot\frac{x-9}{9} - \frac{1}{8}\frac{(x-9)^2}{9^2}\right)$

Then, using the appropriate value for x, approximate $\sqrt{7}$. You may express your answer as a sum.

$\sqrt{7} \approx$ $\frac{143}{54}$

C. Reflect: Is your answer to B. in the right ballpark? (If it is not, perhaps you have not chosen correctly in part A.)

Yes, it's between 2 and 3

5. (8 points) Let us assume that the series $\sum_{n=0}^{\infty} a_n$ converges and $a_n > 0$ for all n. Which of the following series must converge, which must diverge, and for which is it impossible to determine convergence versus divergence? For answers of "must converge" or "must diverge" you must explain your reasoning in order to get full credit. Determine if the following series converge, diverge or if you do not have enough information to determine it. Justify your statements.

(a) $\sum_{n=0}^{\infty}(-1)^n a_n$

$\sum_{n=0}^{\infty} |(-1)^n a_n| = \sum_{n=0}^{\infty} a_n$ Converges. $\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n$ Converges

(b) $\sum_{n=0}^{\infty} \frac{2^n a_n}{3^n}$

$0 < \frac{2^n a_n}{3^n} = \left(\frac{2}{3}\right)^n a_n \leq a_n$. By comparison test.

$\sum_{n=0}^{\infty} \frac{2^n a_n}{3^n}$ Converges.

(c) $\sum_{n=0}^{\infty}(a_n)^{1/3}$ impossible to determine.

① If $a_n = n^{-4}$. $\sum_{n=0}^{\infty} a_n$ Converges. and $\sum_{n=0}^{\infty} a_n^{1/3} = \sum_{n=0}^{\infty} n^{-4/3}$ Converges.

② If $a_n = n^{-2}$. $\sum_{n=0}^{\infty} a_n$ Converges. but $\sum_{n=0}^{\infty} a_n^{1/3} = \sum_{n=0}^{\infty} n^{-2/3}$ diverges

(d) $\sum_{n=0}^{\infty}(a_n)^n$

$n \geq 1$, $(a_n)^n \leq a_n$ ~~for~~ if $0 < a_n < 1$.

Since $\sum_{n=0}^{\infty} a_n$ Converges. there exists an N such that

$n \geq N$. $a_n < 1$. (because $a_n \to 0$, $n \to \infty$)

$$\sum_{n=0}^{\infty}(a_n)^n = \sum_{n=0}^{N}(a_n)^n + \sum_{n=N+1}^{\infty}(a_n)^n$$

$$\leq \sum_{n=0}^{N}(a_n)^n + \sum_{n=N+1}^{\infty} a_n < +\infty \Rightarrow \sum_{n=0}^{\infty}(a_n)^n \text{ Converges}$$

6. (9 points)

A. (2 points) If $f(t)$ and $g(t)$ are both solutions to $x'' + bx' + cx = 0$ and neither f nor g is a constant function, which one of the following must also be a solution to $x'' + bx' + cx = 0$?

(a) $x(t) = 3g(t)f(t)$
(b) $x(t) = e^{f(t)} + e^{g(t)}$
(c) $x(t) = \sqrt{2}f(t) - \frac{g(t)}{3}$
(d) $x(t) = e^{f(t)}$

(c)

$f'' + bf' + cf = 0$

$g'' + bg' + cg = 0$

$\Rightarrow (\sqrt{2}f - \frac{g}{3})'' + b(\sqrt{2}f - \frac{g}{3})' + c(\sqrt{2}f - \frac{g}{3})$

$= \sqrt{2}(f'' + bf' + cf) - \frac{1}{3}(g'' + bg' + cg)$

$= 0.$

B. (7 points) Consider the following differential equations:

$$\begin{aligned} I.\quad & y'' - 4y' + 3y = 0 \\ II.\quad & y'' + 4y' + 3y = 0 \\ III.\quad & y'' + y' + 3y = 0 \\ IV.\quad & y'' + 3y = 0 \end{aligned}$$

(a) For which of the differential equations above is $\lim_{t\to\infty} y(t) = 0$ regardless of the initial conditions? Select all that apply. (There may be more than one answer.)

I: $\lambda^2 - 4\lambda + 3 = 0$, $\lambda_1 = 3$, $\lambda_2 = 1$.

II: $\lambda^2 + 4\lambda + 3 = 0$, $\lambda_1 = -3$, $\lambda_2 = -1$.

III: $\lambda^2 + \lambda + 3 = 0$, $\lambda_{1,2} = \frac{-1 \pm \sqrt{11}\,i}{2}$

IV: $\lambda^2 + 3 = 0$. $\lambda_{1,2} = \pm\sqrt{3}\,i$

Answer: II. III

(b) Which of the differential equations above could model a spring in a frictionless system? Select all that apply.

Answer: IV

(c) Which of the differential equations above (if any) has a family of solutions all of which are periodic with period $\frac{2\pi}{\sqrt{3}}$?

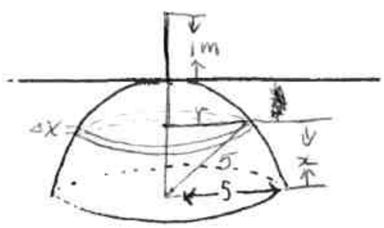
Answer: IV

7. (6 points)

In a new ecologically sound housing development, run-off rainwater is being stored in a hemispherical tank of radius 5 meters, with its flat face 5 meters below the ground and the top of its dome at ground level. (See picture). When the tank is full the residents pump the water out of a hose 1 meter above ground in order to water the garden. Write an integral that gives the work done by the residents in emptying the tank of water? Water has a density of 1000 kg/m^3. You may leave your answer in terms of the gravitational constant g. You need NOT evaluate the integral.

Note: you will have to introduce a variable and integrate with respect to that variable. Indicate clearly where that variable takes the value zero and which direction it increases. Once you have decided this, you must be consistent in your work to get the right answer.

ground level



$$\Delta W = \rho g \cdot \Delta V \cdot (1 + 5 - x)$$

$$= \rho g \, \pi r^2 \cdot \Delta x \, (6 - x)$$

$$r^2 = 5^2 - x^2$$

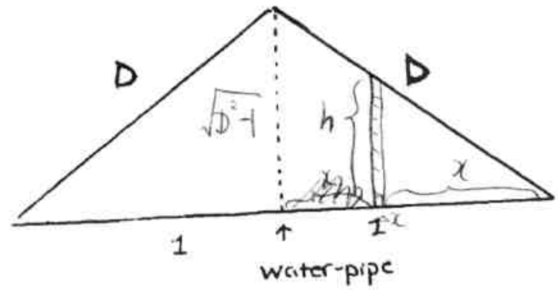
$$= \rho g \cdot \pi (5^2 - x^2) \, \Delta x \, (6 - x).$$

$$\rho = 1000 \ kg/m^3.$$

$$W = \int_0^5 \rho g \, \pi (5^2 - x^2)(6 - x) \, dx$$

$$= \rho g \pi \int_0^5 (5^2 - x^2) \cdot (6 - x) \, dx$$

8. (7 points) A farmer has, through a quirk of inheritance, been bequeathed a field which has the shape of an isosceles triangle where two sides are of length D and one side has length 2. A water-pipe cuts through the field, running from the vertex opposite the side of length 2 to the midpoint of the side of length 2. See the figure below. Assume that the yield density per area depends on the distance m from the water-pipe only and is given by $\rho = \rho(m)$. Write an integral that gives the total yield of the field. Your answer will be in terms of $\rho(m)$ and D.



$$\frac{x}{1} = \frac{h}{\sqrt{D^2-1}}$$

yield of the field of cross section

$$\rho(1-x) \cdot h \cdot \Delta x$$

$\Rightarrow$ The total yield of the field

$$= 2\int_0^1 \rho(1-x) \cdot x\sqrt{D^2-1}\, dx$$

$$= 2\sqrt{D^2-1} \int_0^1 x\,\rho(1-x)\, dx.$$

9. (10 points) Consider the function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Note that $\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = \int_0^\infty e^{-t}\, dt$ and $\Gamma(2) = \int_0^\infty t e^{-t}\, dt$.

(a) Compute $\Gamma(1)$. That is, compute $\int_0^\infty e^{-t}\, dt$.

$$\int_0^\infty e^{-t}\, dt = \lim_{a\to\infty} \int_0^a e^{-t}\, dt = \lim_{a\to\infty} \left(-e^{-t} \Big]_0^a \right)$$

$$= \lim_{a\to\infty} \left(-e^{-a} + 1 \right) = 1$$

(b) Compute $\Gamma(2)$. That is, compute $\int_0^\infty t e^{-t}\, dt$.

$$\int_0^\infty t e^{-t}\, dt \overset{\text{By I.B.P.}}{=\!=\!=} \lim_{a\to\infty} \left(-t e^{-t} \Big]_0^a + \int_0^a e^{-t}\, dt \right)$$

$$= \lim_{a\to\infty} \left(-a e^{-a} - e^{-t} \Big]_0^a \right) = -\lim_{a\to\infty} \frac{a}{e^a} + \lim_{a\to\infty} (-e^{-a} + 1)$$

$$= -\lim_{a\to\infty} \frac{1}{e^a} + 1 = 1$$

(c) You will now show that for any x (≥ 0), if the integral in $\Gamma(x)$ converges, that

$$\Gamma(x+1) = x\Gamma(x).$$

Using integration by parts, show

$$\int_0^\infty t^x e^{-t}\, dt = \int_0^\infty x t^{x-1} e^{-t}\, dt.$$

$$\int_0^\infty t^x e^{-t}\, dt = \lim_{a\to\infty} \left(-e^{-t} \cdot t^x \Big]_0^a + \int_0^a e^{-t} \cdot x t^{x-1}\, dt \right)$$

$$= \lim_{a\to\infty} \left(-e^{-t} \cdot t^x \Big|_0^a \right) + \lim_{a\to\infty} \int_0^a e^{-t} t^{x-1} \cdot x\, dt$$

$$= -\lim_{a\to\infty} \frac{a^x}{e^{+a}} + x \int_0^\infty e^{-t} \cdot t^{x-1}\, dt \quad --- \quad (*)$$

However, $\lim_{a\to\infty} \frac{a^x}{e^a} = \lim_{a\to\infty} \frac{x a^{x-1}}{e^a} = x(x-1) \cdot \lim_{a\to\infty} \frac{a^{x-2}}{e^a} = \cdots$ by L'Hospital's rule, so $\lim_{a\to\infty} \frac{a^x}{e^a} = 0$. We then have $\Gamma(x) = x\Gamma(x-1)$ from $(*)$.

Then, because x does not depend upon t, we can pull the x out of the integral to get $\int_0^\infty t^x e^{-t}\, dt = x \int_0^\infty t^{x-1} e^{-t}\, dt$. This means that for positive integers n, that

$$\Gamma(n) = (n-1)!$$

(In other words, this function, when its domain is restricted to the positive integers, is essentially the factorial function.)

10. (16 points) Consider a habitat in which reside two species: fish and jellyfish. The fish eat the jellyfish and are fished by humans.

Let $F = F(t)$ be the number of thousands of fish at time t.

Let $J = J(t)$ be the number of thousands of jellyfish at time t.

We'll model this situation with the system of differential equations

$$\frac{dF}{dt} = 0.1[\,F(2-F) - \lambda F + FJ]$$
$$\frac{dJ}{dt} = 0.1J(1.5-F)$$

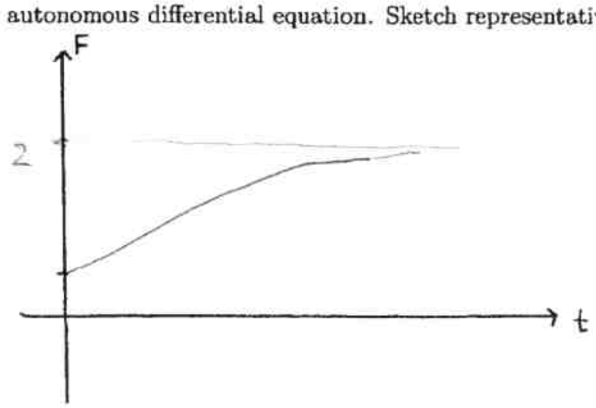
We use the term λF to model the rate at which the fish population is fished by humans.

(a) According to this model, what would happen to the fish population in the absence of both fishing and jellyfish?

$$\frac{dF}{dt} = 0.1F(2-F)$$ Logistic model.

F tends to balance, i.e. $F=0$ or $F \to 2$.

You will have an autonomous differential equation. Sketch representative solutions on the axes below.



$$\int \frac{dF}{F(2-F)} = \int 0.1\, dt$$

$$F = \frac{2Ce^{0.2t}}{1+Ce^{0.2t}}$$

(b) According to this model, what would happen to the fish population in the presence of fishing but the absence of jellyfish?

$$\frac{dF}{dt} = 0.1F(2-F-\lambda)$$

According to this model, do fish have an alternative food source, or do they dine only on jellyfish?

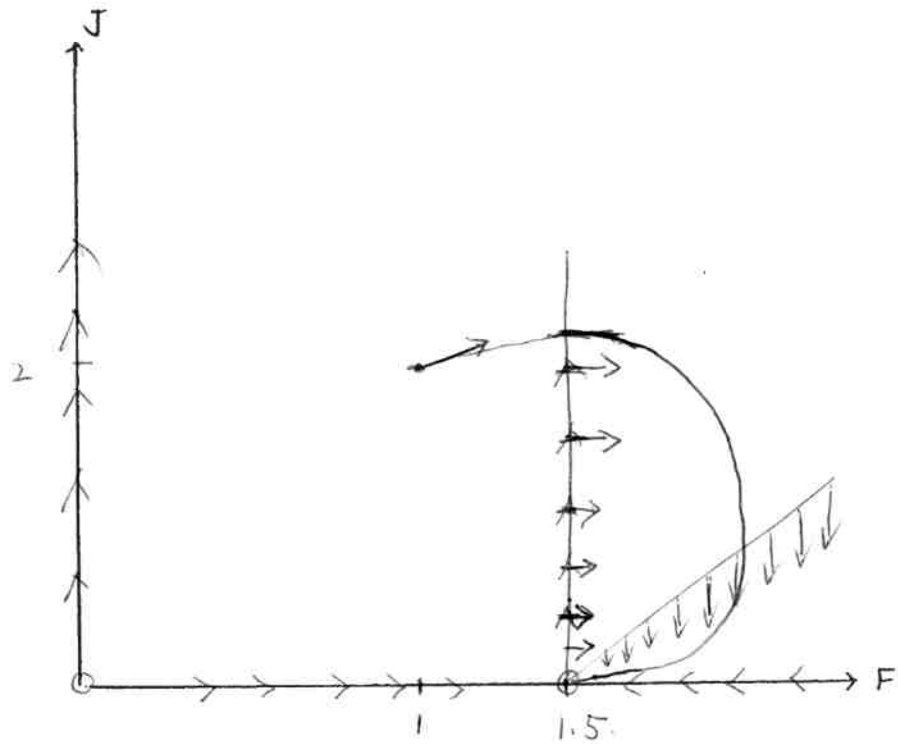
fish has an alternative food source with carrying capacity $2-\lambda$.

(c) According to this model, what would happen to the jellyfish population in the absence of fish?

$$\frac{dJ}{dt} = 0.15J$$, exponentially grow.

(d) Set $\lambda = 1/2$. We rewrite the system on this page. Find the equilibrium points and do a phase plane analysis. Note that we've put J on the vertical axis and F on the horizontal axis. (Make sure your answers are consistent with your answer to (b) and (c) on the previous page.)

$$\frac{dF}{dt} = 0.1[F(2-F) - 0.5F + FJ]$$
$$\frac{dJ}{dt} = 0.1J(1.5-F)$$



$\frac{dF}{dt} = 0$

$0.1F\left[2 - F - 0.5 + J\right] = 0$

$\Rightarrow F = 0$, or $1.5 - F + J = 0$

$\frac{dJ}{dt} = 0$

$J(1.5 - F) = 0$

$\Rightarrow J = 0$ or $1.5 = F$.

Equilibrium points $(0,0)$ and $(1.5, 0)$

(e) If $F(0) = 1$ and $J(0) = 2$, what would happen in the short run? In the long run? Draw the corresponding trajectory in the phase plane above.

F, J will increase in the short run. In the long run Fish will eat all the Jellyfish and tend to equilibrium $F = 1.5$.