Name: ______________________ ID#: ______________

Solutions to Midterm II

Math 1b
Calculus, Series, and Differential Equations

December 1, 2005

Please check your section:

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Rules:

- This is a two-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page.

If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—*Handbook for Students*

1. (15 Points)

(i) *(5 points) Verify that the function* $y = \frac{1}{\sqrt{x}}$ *satisfies the differential equation*

$$\frac{dy}{dx} = -\frac{1}{2x^2y}$$

Solution. All we need to do is check the left- and right-hand sides to the equation.

$$\frac{dy}{dx} = \frac{d}{dx}x^{-1/2} = -\frac{1}{2}x^{-3/2}$$

$$-\frac{1}{2x^2y} = -\frac{1}{2x^2x^{-1/2}} = -\frac{1}{2x^{3/2}}.$$

You can solve this problem explicitly by separation of variables, but we only asked you to verify. ▲

(ii) *(10 points) Find all the possible functions* $y = y(x)$ *which satisfy the differential equation* $y' = xe^{x^2-y}$ *and the initial condition* $y(0) = 0$.

Solution. Rewrite the equation as $e^y\,dy = xe^{x^2}\,dx$. Then

$$\int e^y\,dy = \int xe^{x^2}\,dx$$

$$\implies e^y = \frac{1}{2}e^{x^2} + C.$$

Now if $y(0) = 0$, we have

$$e^0 = \frac{1}{2}e^{0^2} + C \implies C = \frac{1}{2}.$$

So

$$y = \ln\left(\frac{1}{2}e^{x^2} + \frac{1}{2}\right) = \ln\left(e^{x^2} + 1\right) - \ln 2.$$

▲

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2. (10 Points) *Find the general solution to the differential equation*

$$2y' + \frac{2}{x}y - e^x = 0,$$

where $x > 0$.

Solution. This is a first-order linear ordinary differential equation. To put it into standard form, divide by 2 and add the e^x term to the right-hand side.

$$y' + \frac{1}{x}y = \frac{1}{2}e^x.$$

The integrating factor is

$$e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x,$$

so we transform the equation to

$$xy' + y = \frac{1}{2}xe^x.$$

Integrating both sides, we have

$$xy = \int (xy)'\,dx = \frac{1}{2}\int xe^x\,dx = \frac{1}{2}\left(xe^x - e^x\right) + C.$$

$$y = \frac{1}{2}e^x - \frac{e^x}{2x} + \frac{C}{x}.$$

▲

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3. (12 Points) *Match the differential equations with the slope fields graphed below.*

1. ______ $\dfrac{dy}{dx} = 0.1y\,(1 - 0.02y)$

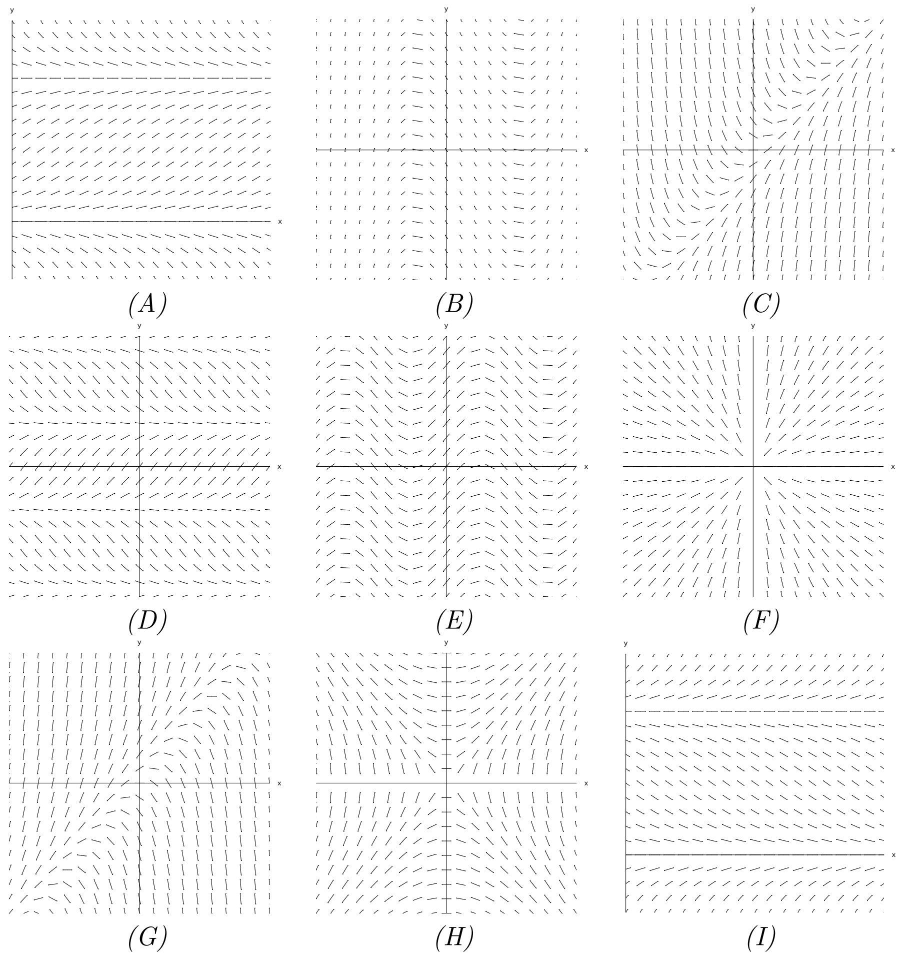
2. ______ $\dfrac{dy}{dx} = x^2 - 2x - 5$

3. ______ $\dfrac{dy}{dx} = \cos y$

4. ______ $\dfrac{dy}{dx} = \dfrac{y}{x}$

5. ______ $\dfrac{dy}{dx} = \dfrac{x}{y}$

6. ______ $\dfrac{dy}{dx} = y - x$



Solution. We can do most of the matching by looking at the places where $\dfrac{dy}{dx} = 0$.

1. $\frac{dy}{dx} = 0$ along two horizontal lines $y = 0$ and $y = 50$. The only slope field with this property is (A).

2. $\frac{dy}{dx} = 0$ along two vertical lines, where x is the two roots of the quadratic on the right-hand side. The only slope field with this property is (B).

3. $\frac{dy}{dx} = 0$ along infinitely many horizontal lines $y = \frac{\pi}{2} + \pi k$, where k is any integer. None of the slope fields have more than two horizontal lines as nullclines except (D).

4. $\frac{dy}{dx} = 0$ where $y = 0$. This is only true in (F).

5. $\frac{dy}{dx} = 0$ where $x = 0$. This is only true in (H).

6. $\frac{dy}{dx} = 0$ where $y = x$, and this is true in (C) and (G). In (G), though, the slope decreases as you go the right—that is, $\frac{dy}{dx}$ gets smaller the bigger x is. This is what we want.

▲

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4. (10 Points) *Wile E. Coyote is planning to blow up the Road Runner at noon next Tuesday and needs one pound of ACME Plutonium as part of his plan. The ACME Plutonium Company makes its deliveries at noon every day, so Wile E. must order his Plutonium in advance.*

At noon on Saturday, Wile E. Coyote takes delivery of one pound of Plutonium. At noon on Sunday, Wile E. checks his supply of Plutonium, and is dismayed to discover that only $\frac{1}{4}$ pound of Plutonium remains. He hadn't realized that Plutonium, like any other radioactive element, decayed exponentially.

Wile E. has one more chance to order Plutonium for his Tuesday "appointment." He needs exactly one pound—any less wouldn't be enough to detonate, and any more would be a waste of money (Plutonium is very expensive!). How much Plutonium should he order to be delivered at noon on Monday?

Solution. Whatever Wile E. has on Monday will be quartered by Tuesday, so he needs to make sure he has 4 pounds of Plutonium on Monday. By that time, his original order will have depleted to $\dfrac{1}{16}$ pounds. So he needs to order

$$4 - \frac{1}{16} = \frac{63}{16}$$

pounds of Plutonium. ▲

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5. (14 Points) *Coca-Cola and Pepsi-Cola are being pumped into the Louvre in Paris. The capacity of the Louvre is 6 megaliters. At $t = 0$, the Louvre is full — it contains 3 megaliters of Coke and 3 megaliters of Pepsi. Coke is pumped in at a rate of 1 megaliter per day, while Pepsi is pumped in at a rate of 2 megaliters per day. The combined mixture overflows out the Louvre's main entrance into the river Seine, at the rate of 3 megaliters per day (so there is no net change in the total amount of liquid in the Louvre).*

Suppose that

$C(t)$ = *amount of Coca-Cola (in megaliters) in the Louvre after t days.*

(a) *(4 points) Write down a differential equation for $C(t)$ that models the above situation, including the initial conditions. Explain what each term in the differential equation means.*

Solution. If $C(t)$ is the number of megaliters of Coke in the Louvre after t days, then

$$\frac{dC}{dt} = (\text{rate of Coke into the Louvre}) - (\text{rate of Coke out of the Louvre})$$

We have

$$\text{rate in} = \frac{1 \text{ ML Coke}}{\text{day}}$$

$$\text{rate out} = \frac{3 \text{ ML solution}}{\text{day}} \times \frac{C(t) \text{ ML Coke in the Louvre}}{6 \text{ ML solution in the Louvre}}$$

So the equation is

$$\frac{dC}{dt} = 1 - \frac{1}{2}C.$$

The initial conditions are $C(0) = 3$. ▲

(b) *(6 points) Solve the differential equation; that is, find an explicit formula for $C(t)$ in terms of t.*

Solution. We have

$$\frac{dC}{dt} = \frac{2-C}{2} \implies \frac{dC}{C-2} = -\frac{1}{2}dt.$$

Integrating, we get

$$\ln(C-2) = -\frac{t}{2} + \text{constant}.$$

Since $C(0) = 3$, we have that this constant must be zero. Hence

$$C - 2 = e^{-t/2} \implies C(t) = e^{-t/2} + 2.$$

▲

(c) *(4 points) What is the eventual (limiting) distribution of Coke and Pepsi in the Louvre? Does your equation for $C(t)$ predict this?*

Solution. The solution predicts that the amount of Coke in the Louvre will be

$$\lim_{t\to\infty}\left(2+e^{-t/2}\right)=2$$

megaliters So there are 4 ML of Pepsi. This does make sense because the fluids pumped in are in the proportion 1 : 2 as well.

The only thing that doesn't make sense is the estimate of 6 ML for the size of the Louvre. A liter is a cubic decimeter, or $\frac{1}{1000}$ of a cubic meter. So six megaliters is 6000 cubic meters. This is the volume of a box 10 meters by 20 meters by 30 meters, which seems a lot smaller than the Louvre.

A Google search will tell you that the floor of the Louvre is 51,615 square meters in area. Supposing the ceiling is a uniform 3 meters high (it isn't–in some galleries it's much higher, such as in the painting below), the volume is the Louvre is more than 154,845 cubic meters. This is more than 154 megaliters. Oh, well.

Projet d'aménagement de la Grande Galerie du Louvre, vers 1789? by Hubert Robert, 1796. Musée du Louvre

▲

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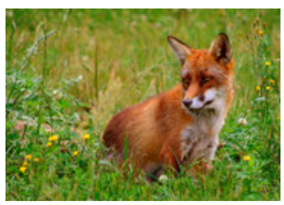
6. (14 Points) *A forest is populated with foxes and pheasants. The two species evolve according to the system of differential equations*

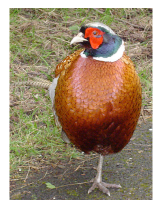
$$y' = xy - y$$
$$x' = x - xy.$$

(a) *(3 points) Which of the functions (x or y) represents the fox population and which the pheasant population?* Explain your answer.

Solution. We apologize for the cultural bias. A pheasant is a bird, hunted by foxes and people. These pictures are not in the same scale—an adult pheasant weighs about 5 pounds.



A fox



A pheasant

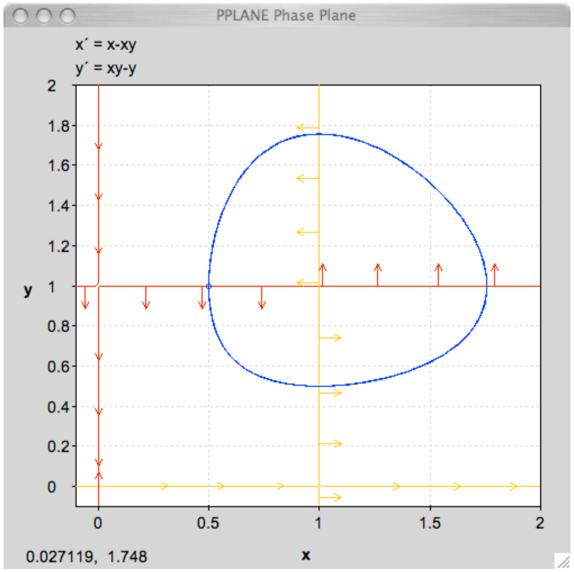
According to the equations, interactions (proportional to xy) between x and y *increase* y and *decrease* x. So x are the foxes and y are the foxes.

Many people reasoned that x was the pheasant population because in the absence of y, x grows exponentially, or something analogous. The observation is true, but it doesn't make x the prey. A different predator-prey system might have $x' = x^3 - xy$, and now the growth of x in the absence of y isn't exponential, it's something else. What makes x the prey is the fact the meeting a fox is detrimental to its growth–that is, the negative xy term. ▲

(b) *(6 points) Draw the phase plane and label all nullclines. Make sure you indicate whether a nullcline is an x-nullcline or y-nullcline. Draw a typical orbit and use arrows to indicate the direction along which it is traversed as time increases.*

Solution. The x-nullclines are the places where $\dfrac{dx}{dt} = 0$. By factoring $xy - y = y(x - 1)$, we see that these are when $y = 0$ (the x-axis) and $x = 1$ (a horizontal line). The y-nullclines are when $x = 0$ (the y-axis) or $y = 1$ (a horizontal line). We allowed you to assume the orbit was closed.

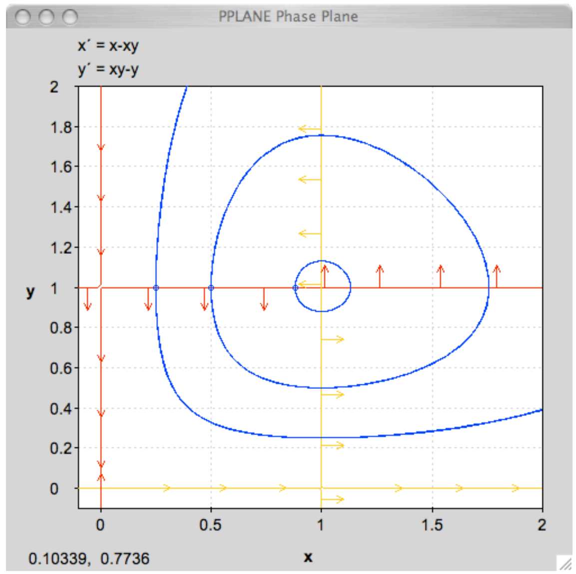
Below is a screenshot from the pplane applet. The x-nullclines are in yellow and the y-nullclines are in red. To go with the flow along the nullclines, the orbit must be traversed in the clockwise direction.



(c) *A town near the forest is becoming overrun with foxes, and the city council proposes a temporary fox hunt to reduce the population of foxes by 50%. An anti-hunting group protests the hunt, saying it would be better to wait until the population was at its* lowest *point. That would result in fewer foxes being killed and would reduce the population to half of its lowest value.*

(i) *(4 points) Indicate graphically what would happen after adopting each group's hunting plans.*

Solution. The fact that the town is being overrun by foxes leads us to indicate that we are on the right-hand side of the orbit. Halving the population at this point would move the orbit closer to the equilibrium point. On the other hand, waiting for the fox population to reach its minimum and *then* halving it will move the orbit further out from the equilibrium point. See the updated phase portrait below.



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(ii) (2 points) Will the anti-hunting group's plan solve the fox problem permanently? Explain your answer.

Solution. The anti-hunting group's plan will result in fewer foxes in the short term. However as time progress we move along the curve to a *much* higher maximum fox population! ▲

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7. (10 Points) *Consider the differential equation*

$$0 = 2y'' - 12y' + 18y.$$

(a) *Write down the characteristic equation and find its root(s).*

Solution. For a trial solution $y = e^{rt}$, the equation becomes

$$0 = r^2 - 6r + 9.$$

(We divide by 2 to make sure the coefficient on r^2 is one). This factors as $(r-3)^2$, so there is only one root: $r = 3$. ▲

(b) *Write down the general solution of this differential equation.*

Solution. Because the single characteristic value is repeated, the general solutions is

$$y = Ae^{3t} + Bte^{3t}.$$

▲

(c) *Find the particular solution to this differential equation that also satisfies the initial conditions*

$$y(0) = 1 \qquad\qquad y'(0) = 1.$$

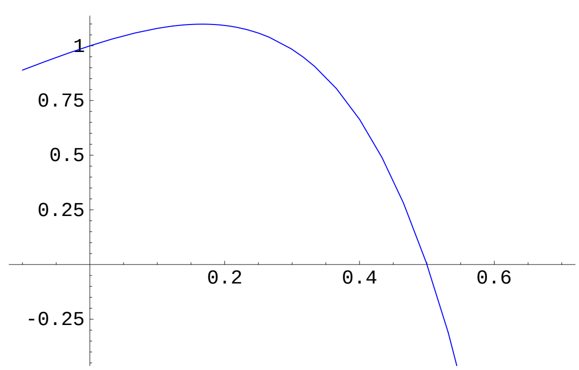
Solution. Plugging in $y(0) = 1$ to the general solution gives $1 = A$. The derivative of y is

$$y' = 3Ae^{3t} + Be^{3t} + 3Bte^{3t},$$

and if $y'(0) = 1$ we have

$$1 = 3A + B = 3 + B \implies B = -2.$$

This means the particular solution is $y = e^{3t} - 2te^{3t}$. ▲



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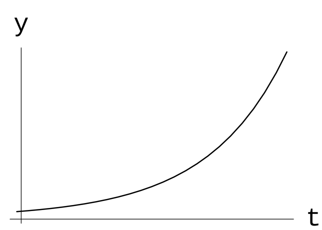
8. (15 Points) *For each of the following graphs of a function $y(t)$, we will find a differential equation of the form $y'' + by' + cy = 0$ so that the function graphed may be a solution of the equation. Do the following for each graph:*

(a) (3 points) Find two numbers that might be roots of the characteristic polynomial.

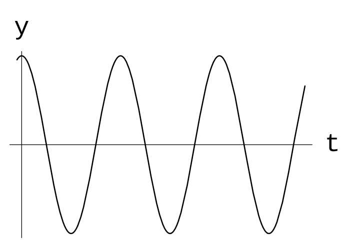
(b) (2 points) Find a polynomial that has roots at the numbers you gave in (a) (Note: If the roots are a and b, the polynomial can be written in the form $(r - a)(r - b)$). Find a differential equation whose characteristic polynomial is this polynomial.

(i)



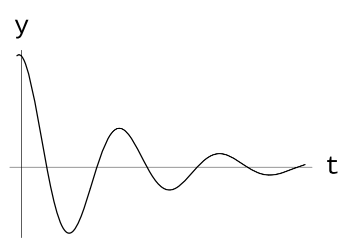
Solution. This could be the graph of a single exponential, so the characteristic polynomial needs two real roots. Let's say 1 and 2. So a reasonable polynomial would be $(r - 1)(r - 2) = r^2 - 3r + 2$. This is the characteristic polynomial of the equation $y'' - 3y' + 2y = 0$. ▲

(ii)



Solution. This could only be the graph of a periodic function like cos. If the roots of the characteristic polynomial are $\alpha \pm \beta i$, the characteristic solutions are $e^{\alpha t} \sin \beta t$ and $e^{\alpha t} \cos \beta t$. So the only chance for true periodicity is $\alpha = 0$, i.e., the roots are purely imaginary, like $\pm i$. These are the roots of $r^2 + 1$, which is the characteristic polynomial of $y'' + y = 0$. ▲

(iii)



Solution. Here there is a decaying oscillation, indicating a complex root with $\alpha < 0$. So possible roots might be $-1 \pm i$. These are the roots of the polynomial $\big(r - (-1 + i)\big)\big(r - (-1 - i)\big) = r^2 + 2r + 2$. So a possible answer is $y'' + 2y' + y = 0$. ▲

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