Name: ______________________________ ID#: ____________________

Solutions to Midterm I

Math 1b
Calculus, Series, and Differential Equations

October 27, 2005

Rules:

- This is a two-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

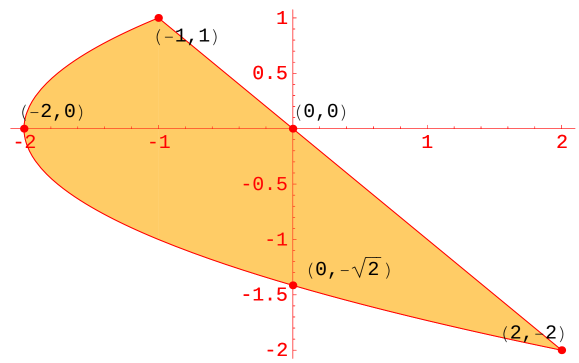
- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—*Handbook for Students*

1. (9 Points) *Let R be the region bounded by $x = y^2 - 2$ and $x = -y$.*

(a) *(5 Points) Draw the region R. Label with x and y coordinates the intercepts (where either curve crosses an axis) and points of intersection.*

Solution.



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(b) *(4 Points) Write down an integral whose value is the area of this region. Draw on the graph a typical approximating rectangle and label its height and width. You need not evaluate the integral.*

Solution. The best way to integrate this is vertically. If we make horizontal strips of height Δy, their contribution to the total area is approximately

$$\Delta A \approx \left(y^2 - 2 - (-y)\right) \Delta y$$

To find the endpoints, we need to find the intersection of the two curves. By solving $y^2 - 2 = -y$, we get $y^2 + y - 2 = 0$, so $y = -2$ and $y = 1$. Therefore

$$A = \int_{-2}^{1} \left(-y - (y^2 - 2)\right)\, dy = -\left[\frac{y^3}{3} + \frac{y^2}{2} - 2y\right]_{-2}^{1} = \frac{10}{3}.$$

▲

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2. (9 Points)

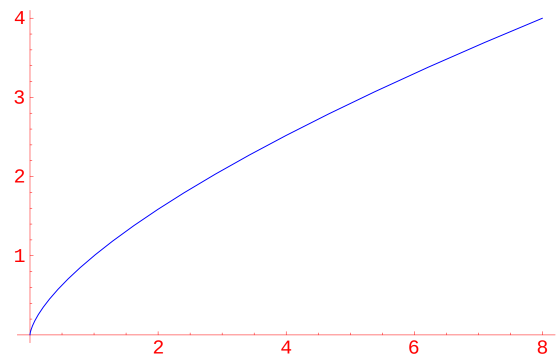
(a) (5 Points) Sketch the graph of the parametric curve

$$x(t) = t^3 \qquad\qquad y(t) = t^2,$$

on the interval $0 \le t \le 2$. *Label with* x *and* y *coordinates the points where* $t = 0$, $t = 1$, *and* $t = 2$.

Hint. Eliminate the parameter and write y as a function of x.

Solution.



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(b) (4 Points) Write down an integral which represents the length of the curve traced out on the interval $1 \le t \le 2$. *Simplify the integral so that the integrand only involves* t *and* dt. *You need not evaluate the integral.*

Solution. We have

$$d\ell = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(3t^2)^2 + (2t)^2}\, dt$$
$$= \sqrt{9t^4 + 4t^2}\, dt = t\sqrt{9t^2 + 4}\, dt.$$

So

$$\ell = \int_1^2 t\sqrt{9t^2+4}\, dt$$
$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right).$$

▲

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3. (12 Points)

(i) (4 Points) Determine whether the integral is convergent or divergent. If it is convergent, evaluate the integral.

$$\int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

Solution. We have

$$\begin{aligned}\int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx &= \lim_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx \\ &= \lim_{b\to\infty} \left[\ln(x-1) - \ln(x+1)\right]_2^b \\ &= \lim_{b\to\infty} \left[\ln \frac{x-1}{x+1}\right]_2^b \\ &= \lim_{b\to\infty} \left[\ln \frac{b-1}{b+1} - \ln \frac{1}{3}\right] \\ &= \ln 1 - \ln \frac{1}{3} = \ln 3.\end{aligned}$$

A common mistake was to split the integral into a difference of two integrals and try to subtract them. The problem is that $\int_2^\infty \frac{dx}{x-1}$ and $\int_2^\infty \frac{dx}{x+1}$ are both divergent, and therefore this difference is indeterminate. ▲

(ii) (4 Points apiece) Use the Comparison Theorem to determine whether these integrals are convergent or divergent. Do not evaluate the convergent integrals.

(1) $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\, dx.$

Solution. This integral converges by comparison to e^{-x}. ▲

(2) $\displaystyle\int_0^1 \frac{1+e^{-x}}{x^3}\, dx.$

Solution. When $0 \le x \le 1$, $1 \ge e^{-x} \ge e^{-1}$. So

$$\frac{1+e^{-x}}{x^3} > \frac{1+e^{-1}}{x^3}$$

The integral of the right-hand side over $(0, 1]$ *diverges* (using the p-test with $p = 3 > 1$), and therefore so must the integral that we were asked about. ▲

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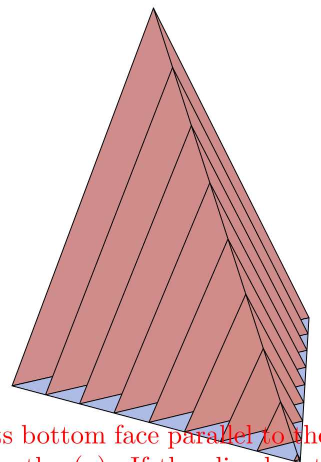
4. (12 Points)

Consider a solid S whose bottom face is an isosceles triangle with base length 1 and "height" $\sqrt{\frac{2}{3}}$. Cross sections of the solid perpendicular to the bottom face and parallel to the base of the triangular bottom are equilateral *triangles. (See the picture.) Find the volume of this solid.*

Hint: *The area of an equilateral triangle of side length s is*

$$A = s^2 \frac{\sqrt{3}}{4}.$$



Solution. Slice the solid perpendicular to its bottom face parallel to the base of that face. The slice at a distance y has length $s(y)$. If the slice has thickness Δy, the volume of this slice is

$$\Delta V \approx s(y)^2 \frac{\sqrt{3}}{4} \Delta y.$$

We need to know $s(y)$. Using similar triangles, we have

$$\frac{\sqrt{2/3}}{1} = \frac{\sqrt{2/3} - y}{s(y)} \implies s(y) = 1 - \sqrt{\frac{3}{2}} y.$$

Therefore the volume is

$$V = \int_0^{\sqrt{2/3}} \left(1 - \sqrt{\frac{3}{2}} y\right)^2 \frac{\sqrt{3}}{4} \, dy = \frac{\sqrt{3}}{4} \sqrt{\frac{2}{3}} \int_0^1 u^2 \, du$$
$$= \frac{\sqrt{2}}{4} \left. \frac{u^3}{3} \right|_0^1 = \frac{\sqrt{2}}{12}.$$

The solid described is the one you would get if you sliced a regular tetrahedron of side length one along planes parallel to one face, and slid all the sliced towards one of the upright faces. Not surprisingly, the volume is the same is that of a regular tetrahedron. ▲

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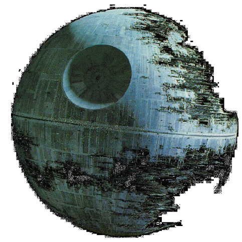
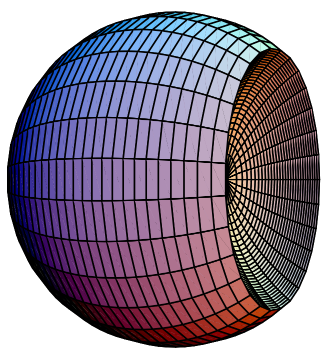
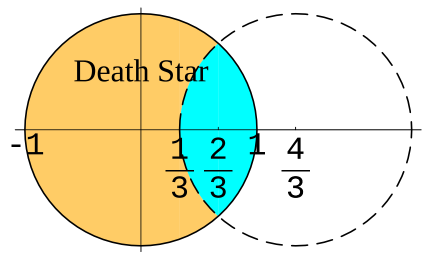
5. (13 Points)



YODA: *Destroy the Death Star you must.*
LUKE: *But how, Master Yoda? It is so … so big.*
YODA: *Very big, yes. But size matters not! A solid of revolution it is. Its precise volume you must find.*
LUKE: *But that's easy. The Death Star is a perfect sphere of radius 1, so its volume is just* $\frac{4}{3}\pi$.
YODA: *(*sighs*) Impatient you are, young Jedi. Much still must you learn.*
LUKE: *I have a bad feeling about this.*
YODA: *A perfect sphere it may* seem. *But closer you must look. Closer, yes. A portion is missing. The missing portion belongs to a sphere, also of radius 1, whose center lies* 4/3 *units from the center of the Death Star. Account for this portion you must.*
LUKE: *I will not fail you … ahem … Fail you I will not, Master.*





Find the volume of the Death Star, as described above.

Solution. If we cut the Death Star through the center, we get the cross-section above. The piece cut out of the sphere has front-to-back symmetry as well as rotational symmetry. Its volume is

$$2\int_{2/3}^{1} \pi(1-x^2)\,dx = 2\pi\left[x-\frac{x^3}{3}\right]_{2/3}^{1} = \frac{16\pi}{81}.$$

Thus the volume left is

$$\frac{4\pi}{3} - \frac{16\pi}{81} = \frac{108\pi}{81} - \frac{16\pi}{81} = \frac{92\pi}{81}.$$

▲

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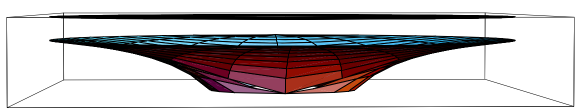
6. (13 Points) *Let R be the region bounded above by the x-axis, below by the curve $y = \frac{1}{x}$ and on the right by the line $x = -1$.*[1]

(i) *(3 Points) Does the region R have finite area? If so, what is it?*

Solution. The area of the region, if it were to exist, would be $\int_{-\infty}^{-1} \frac{dx}{x}$. However, this integral diverges. ▲

(ii) *(5 Points) Let S be the volume generated by revolving R about the y-axis. Draw S and find its volume, if it exists.*

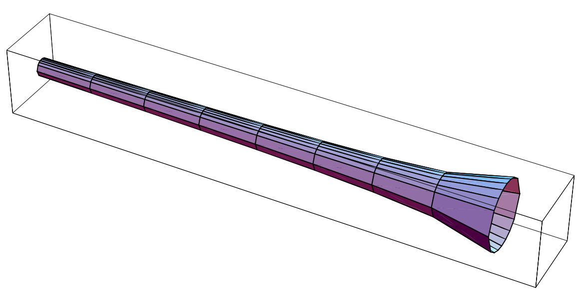


Solution. Using the method of cylindrical shells, that area would be

$$\int_0^1 \frac{2\pi x\, dx}{x^2} = 2\pi \int_0^1 \frac{dx}{x},$$

which we know diverges. ▲

(iii) *(5 Points) Let T be the volume generated by revolving R about the x-axis. (T is sometimes called* Gabriel's Horn.[2] *) Draw T and find its volume, if it exists.*



Solution. Using the disk method, the volume is

$$\int_{-\infty}^{-1} \pi \frac{1}{x^2}\, dx = \pi.$$

▲

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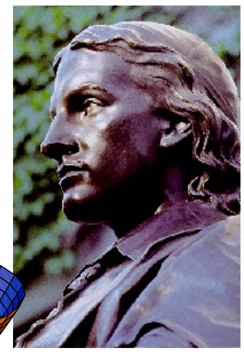
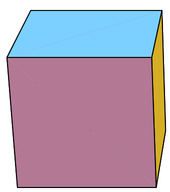
[1]There is a typo fixed here. The original midterm had "$x = 1$."

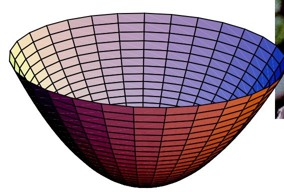
[2]Another typo: This is Gabriel's Horn, not the figure in part (iii). Since this problem was on the Review set and was correctly labeled there, we didn't penalize students who flipped the x and y axes.

7. (18 Points)

John Harvard is throwing a party in Loker Commons, and he needs to prepare by moving a large amount of soda pop from storage into a new container. The storage container is shaped like a cube whose sides have length 1 m. The new container is a bowl 1 m high, shaped like a paraboloid, the solid obtained by rotating the parabola $y = x^2$ around the y-axis $x = 0$.







The cube is filled entirely with soda pop, and he wants to calculate how much work it will take to move it all from the cube into the paraboloid. John Harvard™ brand soda pop has a mass density of $\frac{5000}{9.8}$ kg per cubic meter, so its weight density is 5000 Newtons per cubic meter. Since John Harvard went to college before the invention of calculus, he needs your *help to figure out how much work it will take!*

(a) *(6 Points) Calculate how much work it will take to pump all the soda pop out of the cube from the top.*

Solution. Recall that if force and distance are constant,

$$\text{Work} = \text{Force} \cdot \text{Distance}$$

If the distance moved by various parts of the region varies, we can slice the region into pieces were the distance moved is relatively the same, form a Riemann sum and integrate. In this case, because we are moving the soda pop to the top of the tank, the slabs will horizontal.

Let y be the coordinate describing the height of the slab. Let $y = 0$ be the bottom of the cubical tank and $y = 1$ the top. The work ΔW to move this rectangular slab to the top of the tank will be approximately

$$\begin{aligned} \Delta W \approx \Delta F(1-y) &= \Delta \langle \text{Weight} \rangle (1-y) \\ &= g\Delta \langle \text{mass} \rangle (1-y) \\ &= \rho g \Delta \langle \text{volume} \rangle (1-y) \\ &= \rho \cdot g \cdot 1 \cdot 1 \cdot \Delta y \cdot (1-y). \end{aligned}$$

Here ρ is the mass density of the soda pop and g the gravitational constant. Notice $\rho g = 5000$.

After forming a Riemann sum and taking its limit to make a integral, we get

$$\begin{aligned} W &= \int_0^1 \rho g(1-y)\,dy \\ &= \rho\left[y - \frac{y^2}{2}\right]_0^1 = \frac{\rho g}{2} = 2500 \text{ J} \end{aligned}$$

▲

(b) *(6 Points) Show that the same volume of soda pop put into the bowl will reach a height of* $\sqrt{\dfrac{2}{\pi}}$.

Solution. When the paraboloidal tank is filled to a height b, we can slice it horizontally to find its volume. The incremental volume of a slice is

$$\Delta V = \pi r^2\,\Delta y = \pi x^2\,\Delta y = \pi y\,\Delta y.$$

So the volume of the soda pop in the tank is

$$V = \int_0^b \pi r^2 y\,dy = \frac{\pi y^2}{2}\bigg|_0^b = \frac{\pi b^2}{2} \text{ m}^3.$$

Thus if $V = 1$ (the volume of soda pop in the cubical tank), we must have

$$1 = \frac{\pi b^2}{2} \implies b = \sqrt{\frac{2}{\pi}} \text{ m}.$$

▲

(c) *(6 Points) How much work would it take to pump this volume of soda pop out of the bowl (again, from the top)?*

Solution. As before, we can slice the tank horizontally. The slab at height y moves up to the *top* of the tank. So the incremental work is

$$\begin{aligned} \Delta W &\approx \rho g \Delta V(1-y) \\ &= \rho g \pi y \Delta y(1-y). \end{aligned}$$

So

$$\begin{aligned} W &= \rho g\pi \int_0^b (y-y^2)\,dy = \rho g\pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^b \\ &= \rho g\pi\left[\frac{b^2}{2} - \frac{b^3}{3}\right]_0^b = \rho g\pi\left(\frac{1}{\pi} - \frac{2}{3\pi}\sqrt{\frac{2}{\pi}}\right) \\ &= 5000\left(1 - \frac{2}{3}\sqrt{\frac{2}{\pi}}\right) \text{ J}. \end{aligned}$$

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▲

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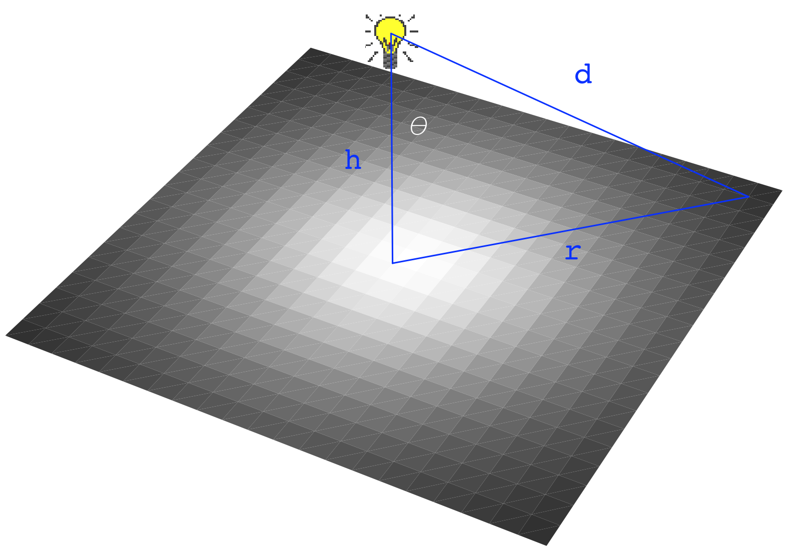
8. (14 Points) *In photometry (the measurement of light), the* illuminance E *of a surface by a point source depends on the intensity* I *of the light, the distance* d *to the surface and the angle of the surface. The law is*

$$E = \frac{I \cos\theta}{d^2},$$

where θ *is the angle between the illuminated surface and the plane perpendicular to the incident light. Illuminance is the "density" of* luminous flux*–the flow of visible light across a surface. If the illuminance across a surface is constant, the rule is*

$$\textit{flux} = \textit{illuminance} \cdot \textit{area}$$

We will use this rule and integration to compute the flux of a light bulb across a plane. Consider the light bulb to be a point source, to have constant intensity I*, and to be held a distance* h *above the origin of the plane.*



(The parts of this problem begin on the next page.)

(a) *(3 Points) Show that the illuminance at a point on the plane a distance r from the origin is*

$$E(r) = \frac{Ih}{(r^2+h^2)^{3/2}}$$

Solution. The illuminance is

$$E = \frac{I\cos\theta}{d^2} = \frac{I\frac{h}{\sqrt{r^2+h^2}}}{r^2+h^2} = \frac{Ih}{(r^2+h^2)^{3/2}}.$$

▲

(b) *(3 Points) Show that the flux across a "thickened circle" of radius r and thickness $r+\Delta r$ is*

$$\Delta F \approx E\,\Delta A \approx 2\pi Ih\frac{r\,\Delta r}{(r^2+h^2)^{3/2}}.$$

Solution. The area of this *annulus* is

$$\Delta A = \pi(r+\Delta r)^2 - \pi r^2 \approx 2\pi r\,\Delta r.$$

Putting the last two together gives the result. ▲

(c) *(1 Point) Write down a Riemann Sum that approximates the total flux across a disk of radius R centered at the origin of the plane.*

Solution. Dividing up the interval $[0, R]$ into n subintervals of width $\Delta r = \frac{R}{n}$, we have an approximation to the total flux given by

$$F \approx \sum_{i=1}^{n} \frac{2\pi Ihr_i^*\,\Delta r}{({r_i^*}^2+h^2)^{3/2}}.$$

▲

(d) *(2 Point) Take the limit of the Riemann Sum–what integral do you get?*

Solution. We have

$$F = 2\pi Ih\int_0^R \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

▲

(e) *(3 Points) Compute this integral.*

Solution. Let $u = r^2 + h^2$. Then $du = 2r\,dr$ and

$$\begin{aligned} F = 2\pi I h \int_0^R \frac{r\,dr}{(r^2+h^2)^{3/2}} &= \pi I h \int_{h^2}^{R^2+h^2} u^{-3/2}\,du \\ &= -2\pi I h\, u^{-1/2}\Big|_{h^2}^{R^2+h^2} \\ &= 2\pi I \left(1 - \frac{h}{\sqrt{R^2+h^2}}\right) \end{aligned}$$

▲

(f) *(2 Points) What is the limit of this expression as $R \to \infty$? Does this value depend on h?*

Solution. The limit as $R \to \infty$ is $2\pi I$, which does not depend on h! ▲

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