Name: ______________________ ID#: ______________

Final Exam

Math 1b
Calculus, Series, Differential Equations

27 January 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please circle your section:

MWF9	Matthew Leingang	TΘ10	Andrew Lobb
MWF10	Ken Chung	TΘ10	Chun-Chun Wu
MWF10	Florian Herzig	TΘ11:30	Amanda Alvine
MWF10	Michael Schein	TΘ11:30	Rosa Sena-Dias
MWF11	Matthew Leingang		
MWF11	Janet Chen		
MWF12	Ken Chung		

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

Problem Number	Possible Points	Points Earned
1	15	
2	9	
3	7	
4	8	
5	10	
6	9	
7	9	
8	19	
9	15	
10	10	
11	10	
12	14	
13	15	
Total	150	

1. (15 Points) Test the following for convergence or divergence. Give justification.

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{17}}$

(ii) $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

(iii) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

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2. (9 Points) Every 5 minutes Mattia (age 10 months) eats 1/3 of the Cheerios on her tray and throws the rest on the floor.[1] Her father gathers the remaining Cheerios off the floor and puts them back on her tray, where she continues to eat a third and toss the rest.

Assume she is given 162 Cheerios at 8:30am. Fill in the following table:[2]

Time	Cheerios on the floor
8:35	
8:40	
8:45	
⋯	⋯
9:30	

[1]Based on a true story.

[2]You may assume that Cheerios are infinitely divisible. This has been experientially verified.

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3. (7 Points)

(a) (2 points) Write down the formula for the sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \cdots = ?$$

For which a and r is the formula valid?

(b) (5 points) Find a power series centered at zero for the function $g(x) = \ln(1 + 3x)$. What is its radius of convergence?

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4. (8 Points) Find a number N such that $S_N = \sum_{n=1}^{N} (-1)^n \frac{1}{n^2}$ is within 10^{-10} of $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$. Make sure you verify the conditions of any theorem you use.

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5. (10 Points) Find $\int e^{2x} \sin 3x \, dx$.

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6. (9 Points) In each of the following, determine a substitution which makes the integral easier. Transform the integral to make it completely in terms of the new variable. You need not carry out the integration any further than this.

(i) $\displaystyle\int_0^1 \frac{e^x\,dx}{e^{2x}+e^{3x}}$

(ii) $\displaystyle\int (u^2+1)\sqrt{u^2+9}\,du$

(iii) $\displaystyle\int \frac{(\ln t)^{46}}{t}\,dt$

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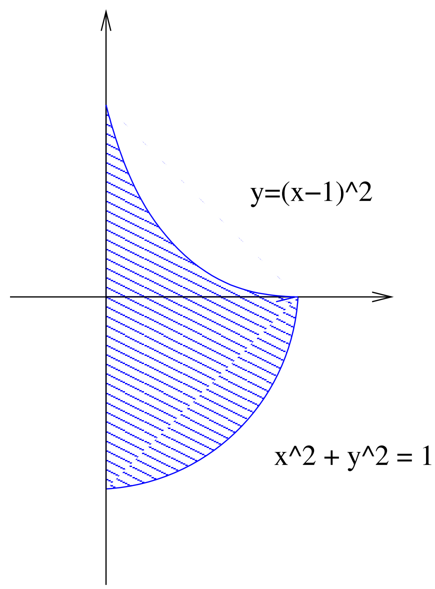
7. (9 Points) Let R be the region bounded by the y-axis, the curve $y = (x-1)^2$, and the curve $x^2 + y^2 = 1$.



What is the volume of the "spinning top" generated by rotating R about the y-axis? (You can check the volume of the bottom half of the solid against a well-known formula. However, your answer should use an integral to prove this answer.)

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8. (19 Points) The city Erauqs has a very homogeneous population. Within its square city limits (4 miles on each side) the population is spread evenly, with constant population density $\rho = 10,500$ people per square mile So the entire city has a population of

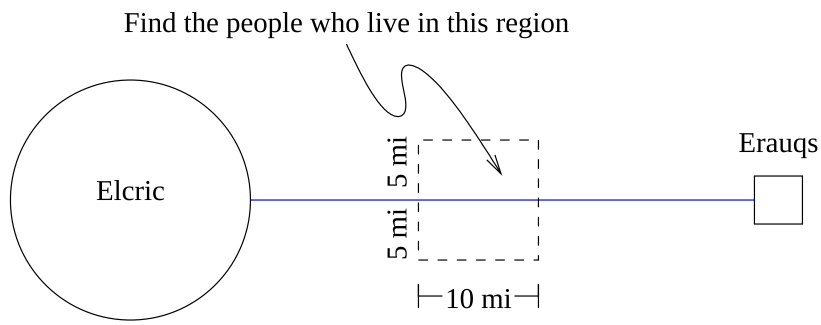
$$(\text{Area})(\text{pop. dens.}) = (16 \text{ sq. mi.})(10,500 \text{ people/sq. mi.}) = 168,000 \text{ people}$$

The nearby metropolis Elcric is more densely populated near the city center. Elcric occupies a circular piece of land with radius 10 miles, and the population density is given by the function $\rho(r) = Ce^{-r^2}$ people per square mile, where r is the distance from the center of Elcric in miles, and $C = \dfrac{8.3 \times 10^7}{\pi(1 - e^{-100})}$ is a constant.

(a) (5 points) Explain why the population of Elcric is $\displaystyle\int_0^{10} 2\pi r C e^{-r^2}\,dr$ (by explaining where each part of the integral comes from).

(b) (5 points) Calculate the population of Elcric.

(c) (9 points) Along a 10 mile stretch, the river running from Erauqs to Elcric runs straight, and the population density at a point x miles away from the river is $\rho(x) = 10(5 - x)$, as long as $x \leq 5$.



How many people live within 5 miles of either side of this 10 mile straight stretch of river?

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9. (15 Points) Solve the initial-value problem

$$y' = 2x(1 + y^2); \qquad y(0) = 1.$$

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10. (10 Points) Crimsonium (Ha on the periodic table) is a very rare radioactive element. Upon enrolling, Fran the Firstyear receives a 100g sample of pure Crimsonium. Upon graduation four years later, however, the amount of Crimsonium in the sample turns out to be only 30g, while the rest has decayed into worthless Bulldogium.

(a) What is the half-life of Crimsonium?

(b) How much Crimsonium will be left for Fran's tenth-year reunion (Note: this is ten years after *graduation*)?

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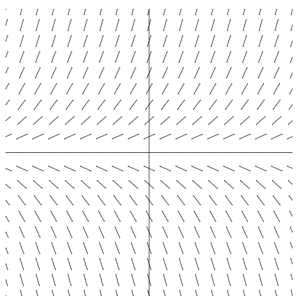
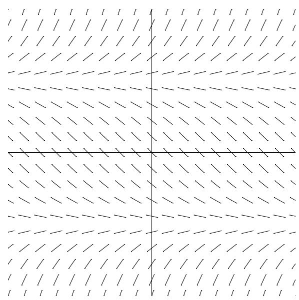
11. (10 Points) Each of the five differential equations below can correspond to only one of the nine direction fields. Determine the matches. (The bounds on the plots are $-2 \le x \le 2$, $-2 \le y \le 2$, but you don't need that information to do the problem).


(A)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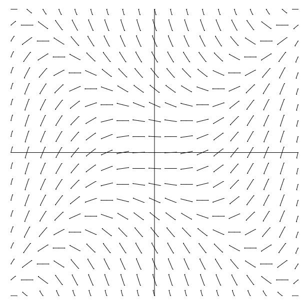
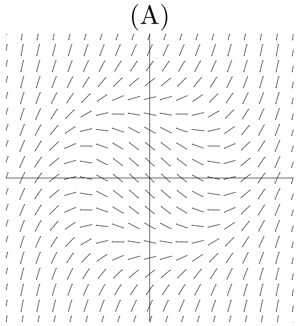

(C)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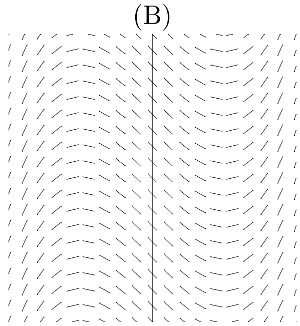
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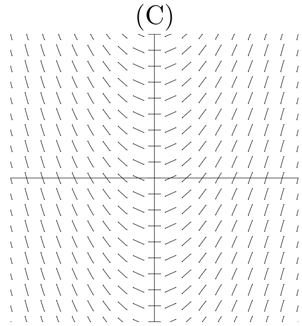
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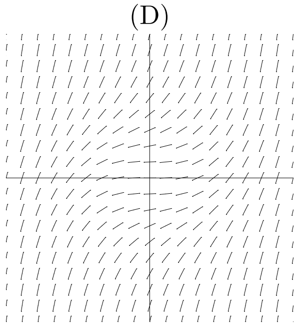
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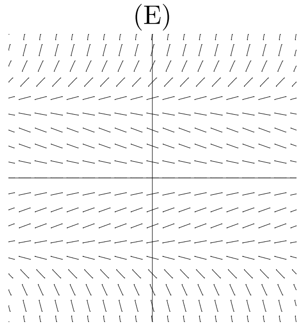
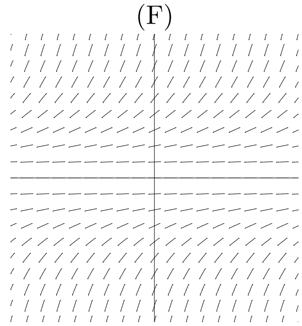

(H)


(I)

______ (i) $y' = x^2 - 1$

______ (ii) $y' = y^2 - 1$

______ (iii) $y' = x^2 - y^2$

______ (iv) $y' = x^2 + y^2$

______ (v) $y' = x^2 + y^2 - 1$

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12. (14 Points) There are two species on Planet Wigglesworth: students and an amazing, living species of pizza slices. Populations of the two species evolve over time according to a predator-prey model:

$$\frac{dx}{dt} = 0.4x - 0.002xy;$$
$$\frac{dy}{dt} = -0.2y + 0.000008xy.$$

(a) (3 points) Which of the variables, x or y, represents the student population and which the pizza slice population? (You may assume that students eat pizza slices and not the other way around!)

(b) (5 points) Find the equilibrium points and explain their significance.

(c) (3 points) Find an expression for $\frac{dy}{dx}$.

(d) (3 points) What happens to y in the absence of x?

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13. (15 Points) Label each of the following statements as true (**T**) or false (**F**). If the statement is true, explain why. If the statement is false, explain why or give an example that disproves the statement.

______ (i) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for all positive p.

______ (ii) $\sum_{n=1}^{\infty} \binom{3/2}{n} 4^{-n} = \frac{1}{2}$.

_____ (iii) $\int_{1/2}^{1} \ln x \, dx = \int_{1/2}^{1} e^y \, dy$

_____ (iv) Suppose the birth rate of a certain country is 3%, the death rate 2% and that 5000 people move out of this country per year (there are no immigrants). If the population of this country is $P(t)$ at year t, then the differential equation for $P(t)$ that describes the above situation is

$$\frac{dP}{dt} = 0.03P - 0.02P - 5000t.$$

_____ (v) The only equilibrium (i.e., constant) solutions to the differential equation $y' = y(y^2 - 1)$ are 0 and 1.

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