

$$16. (a) \frac{x-1}{x^3+x^2} = \frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$(b) \frac{x-1}{x^3+x} = \frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

18. $\frac{x-1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2) + B(x+1)$.
Substituting -2 for x gives $-3 = -B \Leftrightarrow B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2} dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2} \right) dx = [-2 \ln|x+1| + 3 \ln|x+2|]_0^1 \\ = (-2 \ln 2 + 3 \ln 3) - (-2 \ln 1 + 3 \ln 2) = 3 \ln 3 - 5 \ln 2 \quad [\text{or } \ln \frac{27}{32}]$$

24. $\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get
 $x^2-x+6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \Leftrightarrow A = 2$. The coefficients of the x^2 -terms must be equal, so $1 = A+B \Rightarrow B = 1-2 = -1$. The coefficients of the x -terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x} dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3} \right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3} \right) dx \\ = 2 \ln|x| - \frac{1}{2} \ln(x^2+3) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

27.

$$x^2+1 \overline{\begin{array}{r} x^3 \\ x^3+x \\ -x \end{array}}$$

By long division, $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$. Thus,

$$\int_0^1 \frac{x^3}{x^2+1} dx = \int_0^1 x dx - \int_0^1 \frac{x dx}{x^2+1} \\ = \left[\frac{1}{2} x^2 \right]_0^1 - \frac{1}{2} \int_1^2 \frac{1}{u} du \quad [\text{where } u = x^2+1, du = 2x dx] \\ = \frac{1}{2} - \left[\frac{1}{2} \ln u \right]_1^2 = \frac{1}{2} - \frac{1}{2} \ln 2 = \frac{1}{2}(1 - \ln 2)$$

29. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4} dx = \int_3^4 \frac{u}{u^2-4} 2u du = 2 \int_3^4 \frac{u^2}{u^2-4} du = 2 \int_3^4 \left(1 + \frac{4}{u^2-4} \right) du \quad [\text{by long division}] \\ = 2 + 8 \int_3^4 \frac{du}{(u+2)(u-2)}$$

Multiply $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so

$$\frac{1}{(u+2)(u-2)} = \frac{-1/4}{u+2} + \frac{1/4}{u-2} \quad \text{and the last integral is}$$

$$2 + 8 \int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2} \right) du = 2 + 8 \left[-\frac{1}{4} \ln|u+2| + \frac{1}{4} \ln|u-2| \right]_3^4 \\ = 2 + [2 \ln|u-2| - 2 \ln|u+2|]_3^4 = 2 + 2 \left[\ln \left| \frac{u-2}{u+2} \right| \right]_3^4 \\ = 2 + 2 \left(\ln \frac{2}{6} - \ln \frac{1}{5} \right) = 2 + 2 \ln \frac{2/6}{1/5} \\ = 2 + 2 \ln \frac{5}{3} \quad \text{or } 2 + \ln \left(\frac{5}{3} \right)^2 = 2 + \ln \frac{25}{9}$$

Integration Handout A

21. a) $\int x \cos x dx$. Integrate by parts, let $u = x$, $dv = \cos x dx$

b) $\int \cos x \sin^2 x dx$. Integrate by substitution, let $u = \sin x$

c) $\int \frac{x}{x^2 - 4x - 5} dx$. Integrate by partial fractions.

$$\frac{x}{x^2 - 4x - 5} = \frac{A}{(x+1)} + \frac{B}{(x-5)}$$

d) $\int \frac{x-2}{x^2 - 4x + 5} dx$. Integrate by substitution, let $u = x^2 - 4x + 5$

e) $\int \frac{\ln x}{x} dx$. Integrate by substitution, let $u = \ln x$.

f) $\int \ln x dx$. Integrate by parts, let $u = \ln x$, $dv = dx$.

22. $\int \sin^2 \theta d\theta = \frac{1}{2} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta$

$$= \frac{\theta}{2} - \frac{1}{2} \left(\frac{\sin 2\theta}{2} \right) + C$$

$$= \boxed{\frac{\theta}{2} - \frac{\sin 2\theta}{4} + C}$$

23. a) $x = 3 \sin t \Rightarrow dx = 3 \cos t dt$

b) $x = 3 \sin t \Rightarrow \sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 t} = \sqrt{9(1 - \sin^2 t)} = \sqrt{9 \cos^2 t} = 3 \cos t$

c) $0 = 3 \sin t \Rightarrow t = 0$, $3 = 3 \sin t \Rightarrow \sin t = 1 \Rightarrow t = \pi/2$

d) $\int_0^3 \sqrt{9 - x^2} dx = \int_0^{\pi/2} (\sqrt{9 - 9 \sin^2 t} \cdot 3 \cos t) dt$

e) $= \int_0^{\pi/2} 3 \cos t \cdot 3 \cos t dt = 9 \int_0^{\pi/2} \cos^2 t dt = 9 \cdot \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2t) dt$

$$= \frac{9}{2} \left[t \Big|_0^{\pi/2} \right] + \frac{9}{4} \left[\sin 2t \Big|_0^{\pi/2} \right] = \frac{9\pi}{4} + \frac{9}{4} (\sin \pi - \sin 0) = \frac{9\pi}{4}$$

f) $\int_0^3 \sqrt{9 - x^2} dx$ is the area of a quarter circle of radius 3.

Therefore, the area of the entire circle is $A = 4 \int_0^3 \sqrt{9 - x^2} dx$

$$= 4 \left(\frac{9\pi}{4} \right) = 9\pi.$$