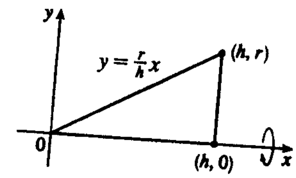
Math 1b Problem Set 4 - Solutions

Section 6.2 # 21, 23, 42

21. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



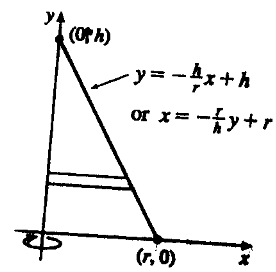
$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\frac{r}{h}y + r$ about the y-axis.

$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \overset{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$

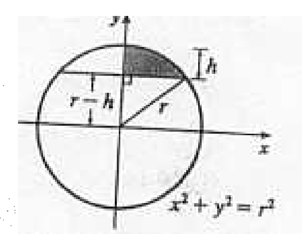
$$= \pi \left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2 h - r^2 h + r^2 h\right) = \tfrac{1}{3}\pi r^2 h$$



* Or use substitution with $u = r - \frac{r}{h}y$ and $du = -\frac{r}{h}\,dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi \frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi \frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

23. $x^2 + y^2 = r^2 \Leftrightarrow x^2 = r^2 - y^2$



$$V = \pi \int_{r-h}^r \left(r^2 - y^2\right) dy = \pi \left[r^2 y - \frac{y^3}{3}\right]_{r-h}^r$$

$$= \pi \left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$$

$$= \pi\left\{\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right]\right\}$$

$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[3r^2 - (r^2 - 2rh + h^2)\right]\right\}$$

$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right]\right\}$$

$$= \tfrac{1}{3}\pi\left(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3\right)$$

$$= \tfrac{1}{3}\pi\left(3rh^2 - h^3\right) = \tfrac{1}{3}\pi h^2(3r - h), \text{ or, equivalently, } \pi h^2\left(r - \frac{h}{3}\right)$$

42. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \Rightarrow r^2 = R^2 - x^2$
$x^2 = R^2 - r^2 \Rightarrow x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x-axis gives us

$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - x^2 - r^2\right) dx \quad \text{[by symmetry]}$$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2 - r^2\right) - x^2\right] dx = 2\pi\left[\left(R^2 - r^2\right)x - \tfrac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}$$

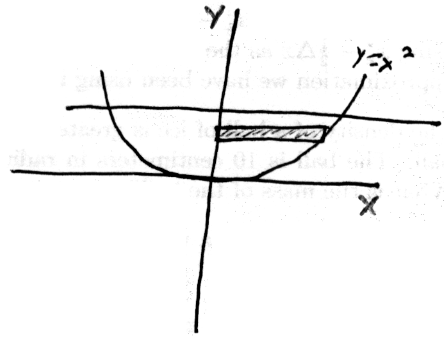
$$= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \tfrac{1}{3}\left(R^2 - r^2\right)^{3/2}\right]$$

$$= 2\pi \cdot \tfrac{2}{3}\left(R^2 - r^2\right)^{3/2} = \tfrac{4\pi}{3}\left(R^2 - r^2\right)^{3/2}$$

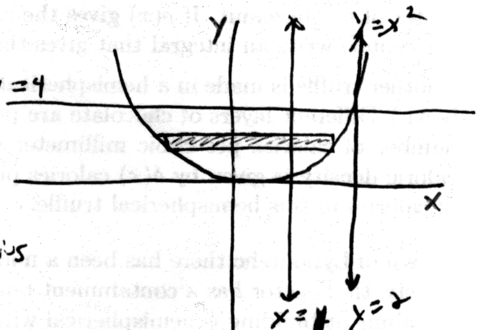
Integration Handout A

#13, 14

13) a. $\pi\int_0^4 (\sqrt{x}-0)^2\,dy = \pi\int_0^4 y\,dy$



b. $\pi\int_0^4 \left[(2\sqrt{y})^2-(2-\sqrt{y})^2\right]dy$

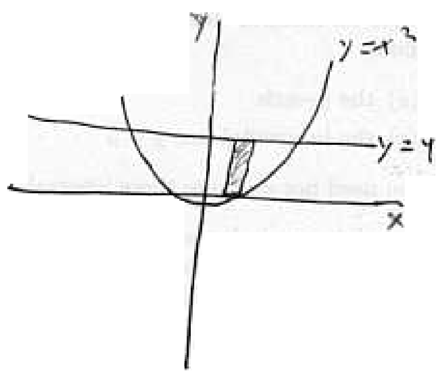


$= \pi\int_0^4 \left[4y-(4-2\sqrt{y}+y)\right]dy$

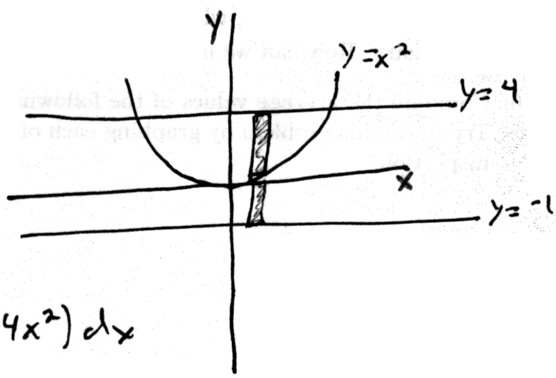
outer radius
$R = 2\sqrt{y}$
inner radius
$r = 2-\sqrt{y}$

$= \pi\int_0^4 (3y+2\sqrt{y}-4)\,dy$

c. $\pi\int_{-2}^{2} (4-x^2)^2\,dx = 2\pi\int_0^2 (4-x^2)^2\,dx$

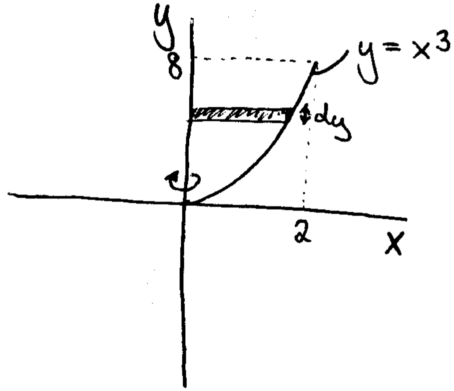


d. outer radius
$R = 4-x^2$
inner radius
$r = x^2-(-1) = x^2+$



$\pi\int_{-2}^{2} \left[(4-x^2)^2-(x^2+1)^2\right]dx = 2\pi\int_0^2 (15-4x^2)\,dx$

14) Volume at height h = $\frac{1}{2}$ × total volume



$$\pi \int_0^h \left(y^{\frac{1}{3}}\right)^2 dy = \frac{\pi}{2}\int_0^8 \left(y^{\frac{1}{3}}\right)^2 dy$$

$$\int_0^h y^{2/3}\, dy = \frac{1}{2}\int_0^8 y^{2/3}\, dy$$

$$\left.\frac{3y^{5/3}}{5}\right|_0^h = \left.\frac{3y^{5/3}}{10}\right|_0^8$$

$$\frac{3h^{5/3}}{5} = \frac{3(8)^{5/3}}{10}$$

$$30h^{5/3} = (15)8^{5/3}$$

$$2h^{5/3} = 8^{5/3}$$

$$2h^{5/3} = 32$$

$$h^{5/3} = 16$$

$$\boxed{h = 5.28}$$