Problem Set 35

1. $y'' = k^2 y$

$$y = c_0 + c_1 x + c_2 x^2 + \dots = \sum_{n=0}^{\infty} c_n x^n$$

$$y'' = 2c_2 + 2\cdot 3 c_3 x + \dots = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$y'' - k^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} k^2 c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - k^2 c_n\right] x^n = 0$$

$$\Rightarrow \quad (n+2)(n+1) c_{n+2} - k^2 c_n = 0$$

$$c_{n+2} = \frac{k^2 c_n}{(n+2)(n+1)}$$

$n=0 \qquad c_2 = \dfrac{k^2 c_0}{2\cdot 1} = \dfrac{k^2 c_0}{2}$

$n=1 \qquad c_3 = \dfrac{k^2 c_1}{3\cdot 2} = \dfrac{k^2 c_1}{6} = \dfrac{k^2 c_1}{3!}$

$n=2 \qquad c_4 = \dfrac{k^2 c_2}{4\cdot 3} = \dfrac{k^2 c_2}{12} = \dfrac{k^4 c_0}{24} = \dfrac{k^4 c_0}{4!}$

$n=3 \qquad c_5 = \dfrac{k^2 c_3}{5\cdot 4} = \dfrac{k^2 c_3}{20} = \dfrac{k^4 c_1}{120} = \dfrac{k_4 c_1}{5!}$

$\vdots$

$$c_{2n} = \frac{k^{2n} c_0}{(2n)!}$$

$$c_{2n+1} = \frac{k^{2n} c_1}{(2n+1)!}$$

$$y = c_0 \sum_{n=0}^{\infty} \frac{k^{2n} x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{k^{2n} x^{2n+1}}{(2n+1)!}$$

2. a. $y' = xy + y + 1$

$$y = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=1}^{\infty} C_{n-1} x^{n-1}$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n$$

$$y' - xy - y = 1$$

$$\sum_{n=0}^{\infty} (n+1) C_{n+1} x^n - x \sum_{n=1}^{\infty} C_{n-1} x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 1$$

$n = 0 \Rightarrow C_1 - C_0 = 1$ because constant terms add up to 1

$$C_1 = C_0 + 1$$

the sum of the non-constant terms is zero

$$\sum_{n=1}^{\infty} [(n+1) C_{n+1} - C_{n-1} - C_n] x^n = 0$$

$$(n+1) C_{n+1} - C_{n-1} - C_n = 0$$

$$C_{n+1} = \frac{C_n + C_{n-1}}{n+1}$$

$n = 1$ $\quad C_2 = \frac{C_1 + C_0}{2} = \frac{C_0 + 1 + C_0}{2} = C_0 + 1/2$

$n = 2$ $\quad C_3 = \frac{C_2 + C_1}{3} = \frac{(C_0 + 1/2) + (C_0 + 1)}{3} = 2/3\, C_0 + 1/2$

$$y = C_0 + (C_0 + 1)x + (C_0 + 1/2)x^2 + (2/3\, C_0 + 1/2)x^3 + \dots$$

b. $y(0) = 0$

$\Rightarrow C_0 = 0$

$y = x + 1/2\, x^2 + 1/2\, x^3 + \dots$

$y(.1) = .1 + 1/2\,(.1)^2 + 1/2\,(.1)^3 + \dots$

$\approx .1055$