Mathematics 1b - Solution Set for PS 3

1 Integration Handout A

8. Set up the integral as per problem 7 (see P. Set #2)

$$\begin{aligned}
\int_0^{10} 2\pi x \rho(x)\, dx &= \int_0^{10} 2\pi x \frac{1010}{\pi(x^2+1)}\, dx \\
&= 2020 \int_0^{10} \frac{x}{x^2+1}\, dx \qquad\qquad (u = x^2+1) \\
&= 1010 \ln(x^2+1)\Big|_0^{10} = 1010 \ln 101 \approx 4661
\end{aligned}$$

10. This is an extension of the two-dimensional slicing of a circle into concentric rings. Here, we cut the sphere into concentric spherical shells, each of which has volume approximately:

$$\begin{aligned}
V_{shell} &= \frac{4}{3}\pi(x+\frac{\Delta x}{2})^3 - \frac{4}{3}\pi(x-\frac{\Delta x}{2})^3 \\
&= \frac{4}{3}\pi\left(x^3 + 3x^2(\frac{\Delta x}{2}) + 3x(\frac{\Delta x}{2})^2 + (\frac{\Delta x}{2})^3 - x^3 + 3x^2(\frac{\Delta x}{2}) - 3x(\frac{\Delta x}{2})^2 + (\frac{\Delta x}{2})^3)\right) \\
&= \frac{4}{3}\pi(6x^2\frac{\Delta x}{2} + 2(\frac{\Delta x}{2})^3)
\end{aligned}$$

As $\Delta x \to 0$, the higher-order term becomes insignificant, and we are left with $4\pi x^2 \Delta x$ as our approximation for the volume of a spherical shell (note that we could have also obtained this approximation by multiplying the surface area of a sphere, $4\pi x^2$, by Δx). Multiplying that by the density function $\rho(x)$, plugging it into the Riemann sum, and taking limits as usual yields the following expression for the mass:

$$\int_0^{10} 4\pi x^2 \rho(x)\, dx$$

Again, note that when $\rho(x) \equiv 1$, the expression reduces to $\frac{4}{3}\pi(10)^3$, the correct answer for that case.

11. (a) This is pretty much the same as problem #10. Using millimeters instead of centimeters make no difference in this case:

$$\int_0^{R} 4\pi x^2 \rho(x)\, dx$$

(b) In this problem, we must slice the hemisphere into "discs" instead of spherical shells. As shown on p. 456 of the textbook, such a disc will have a volume of $\pi(R^2 - x^2)\Delta x$. Thus, multiplying by the density function $\rho(x)$, plugging it into the Riemann sum, and taking limits yet again yields:

$$\int_0^R \pi(R^2 - x^2)\rho(x)\,dx$$

12. (a,b) This is an application of the formula found in problem #11b. Plugging in, the amount of iodine in grams is:

$$\begin{aligned}\int_0^R \pi(R^2 - x^2)\rho(x)\,dx &= \int_0^{100} \pi(100^2 - x^2)(6\times 10^{-5})(200 - x)\,dx \\ &= (6\times 10^{-5})\pi\Big(\int_0^{100} x^3\,dx - 200\int_0^{100} x^2\,dx - 1\times 10^4\int_0^{100} x\,dx + \\ &\qquad 2\times 10^6\int_0^{100} dx\Big) \\ &= 6500\pi \approx 2.0\times 10^4\,g\end{aligned}$$

2 Textbook problems

6.2.28 Represent the circle with the equation $x^2 + y^2 = r^2$. A quick sketch shows that a square-shaped cross section of the solid has side $2\sqrt{r^2 - x^2}$ and thus area $4(r^2 - x^2)$. The volume is then:

$$\int_{-r}^{r} 4(r^2 - x^2)\,dx = 8\int_0^r (r^2 - x^2)\,dx = 8(r^3 - \frac{1}{3}r^3) = \frac{16}{3}r^3$$

6.2.37 (a) The expression for volume, as given on p. 455 of the textbook, is:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

The integral only depends on the function $A(x)$. If a family of parallel planes gives equal cross-sectional areas for two different solids, the resulting $A(x)$'s are exactly the same for the two solids. Thus, the integrals, and the volumes, are equal.

(b) The horizontal cross sections of the oblique cylinder have the same area as the horizontal cross sections of a normal cylinder. Thus, by Cavalieri's Principle, the volumes are the same, i.e., the volume $= \pi r^2 h$.