Problem Set 28

7.3: #4, 10, 16,
7.5: #7
Handout: 6, 7, 8, 9

Differential Equations Handout A

$y'' + 9y = 0$

a) $y'' = 9e^{3t} + 9e^{-3t}$

$9e^{3t} + 9e^{-3t} + 9(e^{3t} + e^{-3t}) \neq 0$

<u>NO</u>

b) $y'' = ce^{t}$

$ce^{t} + 9(ce^{t} - t) \neq 0$

<u>NO</u>

$y'' = 2c$

$2c + 9(c(t^2 + t)) \neq 0$

<u>NO</u>

d) $y'' = -9\sin 3t$

$-9\sin 3t + 9(\sin 3t + 6) \neq 0$

<u>NO</u>

e) $y'' = -9 \cdot 5\cos 3t$

$-9 \cdot 5\cos 3t + 9 \cdot 5\cos 3t = 0$

<u>Yes</u>

7. $y' = y + 1$

a) $y' = Ce^{t}$

$Ce^{t} \neq Ce^{t} + 1$

<u>NO</u>

b) $y' = Ce^{t} - 1$

$Ce^{t} - 1 \neq Ce^{t} - t + 1$

<u>NO</u>

c) $2Ct + C \neq Ct^2 + Ct + 1$

<u>NO</u>

d) $Ce^{t} = Ce^{t} - 1 + 1$

<u>Yes</u>

e) $-Ce^{-t} = Ce^{-t} + 1 + 1$

<u>NO</u>

a) $\frac{dM}{dt}$ kM $\frac{dM}{dt}$ 250 when M 5000

$250 \quad 5000k \Rightarrow k \quad 1/20 \qquad \frac{dM}{dt} \; \frac{1}{20} M, \; M \; 1000$

$\frac{dB}{dt} = kB$ $\Rightarrow$ Solution has the form $B(t) = C\,e^{kt}$

$B(0) = C \cdot e^{k \cdot 0} = 600 \Rightarrow C = 600$

$B(10) = 600\, e^{k \cdot 10} = 800 \Rightarrow k = \frac{1}{10} \ln(4/3)$

So, $\boxed{\frac{dB}{dt} = \frac{1}{10} \ln(4/3)\, B, \quad B(t) = 600\, e^{1/10 \ln(4/3) t}}$

a) $\frac{dP}{dt} = .03P - 6000$

$\frac{dP}{dt} - .03(P \quad 200{,}000)$

$\int \frac{dP}{P \quad 200{,}000} \quad \int 03t$

$\ln P \quad 200{,}000 \qquad 03t + C$

$P \quad 200{,}000 \quad Ce^{03t}$

$P(t) - Ce^{.03t} + 200\,000$

$P(0) = Ce^{-} + 200000 = 3{,}000{,}000$

$C = 2{,}800{,}000$

$\boxed{P(t) = 2{,}800{,}000\, e^{.03t} + 200{,}000}$

4. $y' = xy \Rightarrow \int \frac{dy}{y} = \int x\,dx \quad [y \neq 0] \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow |y| = e^C e^{x^2/2} \Rightarrow y = Ke^{x^2/2}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

10. $\frac{dy}{dx} = \frac{y \cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \Rightarrow \frac{1+y^2}{y}\,dy = \cos x\,dx \Rightarrow$ $\int \left(\frac{1}{y} + y\right) dy = \int \cos x\,dx \Rightarrow \ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow$ $C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

16. $\frac{dy}{dx} = \frac{y^2}{x^3}$, $y(1) = 1$. $\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + C$. $y(1) = 1 \Rightarrow -1 = -\frac{1}{2} + C \Rightarrow$ $C = -\frac{1}{2}$. So $\frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{2+2x^2}{2 \cdot 2x^2} \Rightarrow y = \frac{2x^2}{x^2+1}$.

29. $\frac{dP}{dt} = k(M-P) \Leftrightarrow \int \frac{dP}{P-M} = \int (-k)\,dt \Leftrightarrow \ln|P-M| = -kt + C \Leftrightarrow |P-M| = e^{-kt+C}$ $\Leftrightarrow P - M = Ae^{-kt}$ $[A = \pm e^C] \Leftrightarrow P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then $P(0) = 0 \Leftrightarrow 0 = M + A \Leftrightarrow A = -M \Leftrightarrow P(t) = M - Me^{-kt}$. $\lim_{t\to\infty} P(t) = M - M \cdot 0 = M$.

7. **(a)** Our assumption is that $\frac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, we substitute $y = \frac{P}{K}$, $P = Ky$, and $\frac{dP}{dt} = K\frac{dy}{dt}$, to obtain $K\frac{dy}{dt} = k(Ky)(1-y) \Leftrightarrow \frac{dy}{dt} = ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t) = \frac{K}{1+Ae^{-kt}}$, where $A = \frac{K-P_0}{P_0}$. We use the same substitution to obtain $Ky = \frac{K}{1 + \frac{K-Ky_0}{Ky_0}e^{-kt}}$

y_0

$\frac{1}{2} = y(4) = \frac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so $y = \frac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \frac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get $2y + 23y\left(\frac{2}{23}\right)^{t/4} = \frac{2-2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23} \cdot \frac{1-y}{y} \Rightarrow \left(\frac{2}{23}\right)^{t/4-1} = \frac{1-y}{y}$. It follows that $\frac{t}{4} - 1 = \frac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \frac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$. When $y = 0.9$, $\frac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \frac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.