PROBLEM SET 24

8.5

23. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}}\cdot\frac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0 \text{ for}$$

all x. So $J_1(x)$ converges for all x and its domain is $(-\infty,\infty)$.

38. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since $c_2 = \dfrac{1}{1792} < 0.001$ we use $\displaystyle\sum_{n=0}^{1}\frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24}-\frac{1}{160}\approx 0.0354.$

23. Let $s(t)$ be the position function of the car, and for convenience set $s(0)=0$. The velocity of the car is $v(t)=s'(t)$ and the acceleration is $a(t)=s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0)+v(0)t+\frac{a(0)}{2}t^2 = 20t+t^2$. We estimate the distance travelled during the next second to be $s(1)\approx T_2(1) = 20+1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s $\approx$ 313 mi/h!)

(extra credit) **25.** $E = \dfrac{q}{D^2}-\dfrac{q}{(D+d)^2} = \dfrac{q}{D^2}-\dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 \quad \left(1+\dfrac{d}{D}\right)^{-2}\right]$

We use the Binomial Series to expand $(1+d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1-\left(1-2\left(\frac{d}{D}\right)+\frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 \quad \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3+\cdot\right)\right]$$

$$= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right)-3\left(\frac{d}{D}\right)^2+4\left(\frac{d}{D}\right)^3 \quad\right] \approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

Concept Check

3. (a) A geometric series has some r where $r = \frac{a_{n+1}}{a_n}$ for all n. An infinite geometric series converges when $|r|<1$ and converges to $S = \frac{a}{1-r}$

(b) A p-series is in the form $\sum\frac{1}{x^p}$ It converges when $p>1$.

4. If $\sum a_n = 3$, then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$.

10. (a) $T_n(x) = \displaystyle\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ [$a=0$ in part (b)]

(d) See Theorem 8.7.8.

(e) See Taylor's Inequality (8.7.9).

True false

· see next page.

11. (a) $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdot$ $\qquad(-1,1)$

(b) $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+$ $\qquad(-\infty,\infty)$

(c) $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+$ $\qquad(-\infty,\infty)$

(d) $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdot\cdot$ $\qquad(-\infty,\infty)$

$\tan^{-1}x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdot\cdot$ $\qquad[-1,1]$

TRUE-FALSE

1. False. See Note 2 after Theorem 8.2.6.
2. True by Theorem 8.5.3.
   Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.
3. False. For example, take $c_n = (-1)^n / (n6^n)$.
4. True by Theorem 8.5.3.
5. False, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^3}{(n+1)^3}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1$.
6. True, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n!}{(n+1)!}\right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1$.
7. False. See the note after Example 4 in Section 8.3.
8. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$.
9. True. See (6) in Section 8.1.
10. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 8.4.1.
11. True. By Theorem 8.7.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \Rightarrow f'''(0) = 2$.
    Or: Use Theorem 8.6.2 to differentiate f three times.
12. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.
13. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.
14. True by Theorem 8.1.7 (the Monotonic Sequence Theorem), since $\{a_n\}$ is decreasing and $0 < a_n \leq a_1$ for all n $\Rightarrow$ $\{a_n\}$ is bounded.
15. True by Theorem 8.4.1. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]
16. True. $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ [Theorem 8.2.6].
17. False. The Integral Test tells us that the series $\sum_{n=1}^{\infty} a_n$ also converges, but its value is not equal to the value of $\int_1^{\infty} f(x)\,dx$. In fact, a picture like Figure 2 on page 584 shows that the sum of the series is larger than the value of the integral.

Series Handout A

25. (a) A sum cannot be given because this geometric series diverges. It has $r = (-1)$, which is not included in the interval of convergence $-1 < r < 1$.

(b)

$$\begin{array}{rl} S = & 3 + 3\cdot 7^2 + 3\cdot 7^4 + \cdots + 3\cdot 7^{20} \\ - \quad (7^2)\,S = & \qquad 3\cdot 7^2 + 3\cdot 7^4 + \cdots + 3\cdot 7^{20} + 3\cdot 7^{22} \\ \hline (1-7^2)\,S = & 3 - 3\cdot 7^{22} \end{array}$$

$$S = \frac{3\,(1-7^{22})}{1-7^2}$$

Charlie's closed form formula, $\dfrac{a(1-r^{n+1})}{1-r}$, is correct. However, he calculated n wrong. The correct expression for the sum is $S = \dfrac{3(1-49^{11})}{1-49}$.

Charlie's statement about convergence is not correct. If this series were infinite, it would indeed diverge. But the series is finite, so it has a definite sum (that we calculated above).

(c) Charlie is incorrect. The Harmonic Series has $\lim_{n\to\infty} a_n$ and diverges. Amanda is referring to the n^{th} term test for divergence, which states that if $\lim_{n\to\infty} a_n \neq 0$, the series diverges. For $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.