PROBLEM SET 23

8.7

34. $\int e^{x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}$

8.8

1. The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 +$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)x^n}{2^n \cdot n!} \text{ for } |x| < 1, \text{ so } R = 1\end{aligned}$$

9. (a) $1/\sqrt{1-x^2} = \left[1 + (-x^2)\right]^{-1/2}$

$$\begin{aligned}&= 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}\end{aligned}$$

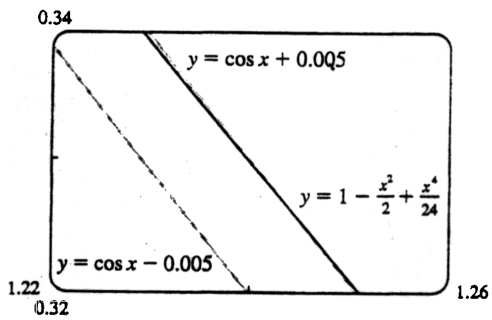
(b) $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$

$= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$ since $0 = \sin^{-1} 0 = C$.

8.9

20. Example 6 in Section 8.6 gives the Maclaurin series for $\ln(1+x)$ as $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$. Thus, $\ln 1.4 = \ln[1 - (-0.4)] = -\sum_{n=1}^{\infty} \frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

22. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \;\Leftrightarrow\; x^6 < 720(0.005) \;\Leftrightarrow\; |x| < (3.6)^{1/6} \approx 1.238$. The curves intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



8.6

6. $f(x) = \frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=0}^{\infty} (-1)^n 3^{2n} x^{2n}$. The series converges when $\left|-9x^2\right| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

16. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Thus,

$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1}$ for $\left|\frac{x}{3}\right| < 1 \;\Leftrightarrow\; |x| < 3$, so $R = 3$.

Series Handout A

24. (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +$$

To get the 3rd degree polynomial for $f(x) = \sin x \cos x$, we only need to multiply the first two terms of the Taylor polynomials of $\sin x$ and $\cos x$.

$$\sin x \cos x \approx \left(x - \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2!}\right) = x - \frac{x^3}{2!} - \frac{x^3}{3!} \left[+ \frac{x^5}{3!2!}\right.$$

exclude because its degree is greater than 3.

$$= x - \frac{x^3}{2} - \frac{x^3}{6} = \boxed{x - \frac{2}{3}x^3}$$

Check:

$\sin(2x) = 2 \sin x \cos x$

$\frac{\sin(2x)}{2} = \sin x \cos x$

- substitute $2x$ for x in the Taylor polynomial for $\sin x$, up to the 3rd degree term

$$\frac{\sin(2x)}{2} = \frac{(2x) - \frac{(2x)^3}{3!}}{2} = \boxed{x - \frac{2}{3}x^3}$$

b) $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5 \cdots (2n-3)}{2^n n!} x^{3n+1}$

To get the 2nd degree polynomial for $f(x) = \sqrt{1+x}\cdot\sqrt{1+x}$ we only need to multiply the 1st 3 terms

$$\Rightarrow \sqrt{1+x}\cdot\sqrt{1+x} = \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)\cdot\left(1 + \frac{x}{2} - \frac{x^2}{8}\right)$$

$$= \boxed{1 + x}$$ (excluding all terms with degree greater than 2)

- The x^3 coefficient should be 0 (remember that our function is just $1+x$)