Math 16: Problem Set 22

#21 on Series Handout A

a) Radius of convergence $R = \frac{1}{2}(7-(-5)) = \frac{1}{2}(12) = 6.$

$$\left.\begin{array}{l} a-6=-5 \\ a+6=7 \end{array}\right\} \boxed{a=1}$$

b) Since $x = 6.5$ is contained in the interval of convergence,

the series converges for $x = 6.5$.

Since $x = -6.5$ is not contained in the interval of convergence,

the series does not converge for $x = -6.5$.

#22 on Series Handout A

a) Recall: $\frac{d}{du}[\ln(1+u)] = \frac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 - u^5 + \cdots$

Integrate term-by-term the Maclaurin series for $\frac{1}{1+u}$ to get

the Maclaurin series for $\ln(1+u)$:

$$u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}}$$

b) Letting $u = x-1$, we have

$$\ln(1+u) = \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}}$$

c) $f(x) = \ln x, \quad a = 1$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$ for $n \geq 1$.

Taylor series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\,1^n}(x-1)^n$

$= \frac{\ln 1}{1}(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$

This answer is identical to that obtained in part 1b).

#23 on Series Handout A

$f(x) = x^{1/3}$

a) The first three derivatives of $f(x)$ are $f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

$f'''(x) = \frac{10}{27}x^{-8/3}$

At $a = 27$, $T_3(x) = f(27) + f'(27)(x-27) + \frac{1}{2}f''(27)(x-27)^2 + \frac{1}{6}f'''(27)(x-27)^3$

$$T_3(x) = 3 + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{9}\right)(x-27) - \left(\tfrac{1}{2}\right)\left(\tfrac{2}{9}\right)\left(\tfrac{1}{243}\right)(x-27)^2 + \left(\tfrac{1}{6}\right)\left(\tfrac{10}{27}\right)\left(\tfrac{1}{6561}\right)(x-27)^3$$

$$= \boxed{3 + \tfrac{1}{27}(x-27) - \tfrac{1}{2187}(x-27)^2 + \tfrac{5}{531441}(x-27)^3}$$

b) $T_3(28) = 3 + \frac{1}{27} - \frac{1}{2187} + \frac{5}{531441} = \boxed{\frac{1613768}{531441} \approx 3.036589198 \approx \sqrt[3]{28}}$

c) An upper bound for the error in this approximation is

$$E = |a_4| = \left|\left(\tfrac{1}{24}\right)\left(-\tfrac{8}{3}\right)\left(\tfrac{10}{27}\right)(27)^{-11/3}(1)\right|$$

$= \left| \frac{1}{6!} \right.$ coefficient of derivative $\left. (28-27)^4 \right|$

$$= \boxed{\frac{10}{43046721} \approx 0.0000002321}$$

d) Now, using Taylor's Inequality, we have error

$E = \frac{M}{(n+1)!}|x-a|^{n+1}$ where $n = 3$ and $|f^{(n+1)}(x)| \leq M$.

Since $f^{(k+1)}(28) < f^{(k)}(28)$ for all $k \geq n+1$, let

$$M = |f^{(n+1)}(x)| = |f^{(4)}(27)| = \left(\tfrac{8}{3}\right)\left(\tfrac{10}{27}\right)27^{-11/3} = \frac{80}{14348907}$$

$$E = \frac{M}{4!}|28-27|^4 = \frac{M}{24} = \boxed{\frac{10}{43046721} \approx 0.0000002321}$$

11.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3\cdot 2x^{-4}$	$-3\cdot 2$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\vdots$	$\vdots$	$\vdots$

So $f^{(n)}(1) = (-1)^n n!$, and $\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$. If $a_n = (-1)^n (x-1)^n$ then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-1| < 1$ for convergence, so $R = 1$.

23. $\sin^2 x = \frac{1}{2}[1-\cos 2x] = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right] = 2^{-1}\left[1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$, $R = \infty$

29. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2)^5 + \frac{1}{6!}(0.2)^6 - \cdots$

But $\frac{1}{6!}(0.2)^6 = 8.\overline{8} \times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!} \approx 0.81873$, correct to five decimal places.

36. $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$, so

$\int_0^{0.5} \cos(x^2)\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^{0.5} = 0.5 - \frac{(0.5)^5}{5\cdot 2!} + \frac{(0.5)^9}{9\cdot 4!} - \cdots$, but $\frac{(0.5)^9}{9\cdot 4!} \approx 0.000009$, so by the Alternating Series Estimation Theorem, $\int_0^{0.5} \cos(x^2)\,dx \approx 0.5 - \frac{(0.5)^5}{5\cdot 2!} \approx 0.497$ (correct to three decimal places).