Math 16: Problem Set 21

#20 on Series Handout A

A power series of the form $\sum_{k=0}^{\infty} c_k (x-2)^k$ is centered at $x=2$.

A radius of convergence of 7 leaves four possibilities for the interval of convergence, depending on the end points:

① $[-5, 9]$, the series converges at both endpoints

② $[-5, 9)$, the series converges at $x=-5$ but not at $x=9$

③ $(-5, 9]$, the series converges at $x=9$ but not at $x=-5$

④ $(-5, 9)$, the series converges at neither endpoint.

4 If $a_n = \frac{(-1)^n x^n}{\sqrt{n+1}}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim_{n\to\infty} \frac{|x|}{\sqrt{1+1/n}} = |x|$
By the Ratio Test, the series $\sum \frac{x^n}{\sqrt{n}}$ converges when $|x|<1$, so the radius of convergence $R=1$. When $x=1$, the series $\sum \frac{1}{\sqrt{n}}$ diverges because it is a p-series with $p=\frac{1}{2}<1$. When $x=-1$, the series $\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence $I=[-1,1)$.

12. If $a_n = n^3(x-5)^n$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3|x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5|<1 \Leftrightarrow -1<x-5<1 \Leftrightarrow 4<x<6$. When $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x=6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges by the Test for Divergence. Thus, $R=1$ and $I=(4,6)$.

14. If $a_n = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+1)(2n)} = 0<1$ for all x. By the Ratio Test the series converges for all x, so $R=\infty$ and $I=(-\infty,\infty)$.

16. If $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3|<1 \Leftrightarrow$$

$|x+3|<\frac{1}{2}$ [so $R=\frac{1}{2}$] $\Leftrightarrow -\frac{7}{2}<x<-\frac{5}{2}$. When $x=-\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p=\frac{1}{2}\le 1$. When $x=-\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I=\left(-\frac{7}{2},-\frac{5}{2}\right]$.

20. We are given that the power series $\sum_{n=0}^{\infty}c_n x^n$ is convergent for $x=-4$ and divergent when $x=6$. So by Theorem 3 it converges for at least $-4\le x<4$ and diverges for at least $x\ge 6$ and $x<-6$. Therefore:

(a) It converges when $x=1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x=8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x=-3$; that is, $\sum c_n(-3)^n$ is convergent.

(d) It diverges when $x=-9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

11. (a) $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = -\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(a)] $= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R=1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2}=(-1)^n$].

(b) $f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R=1$.

(c) $f(x) = \frac{x^2}{(1+x)^3} = x^2\cdot\frac{1}{(1+x)^3} = x^2\cdot\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.