Mathematics 1b - Solution Set for PS 2

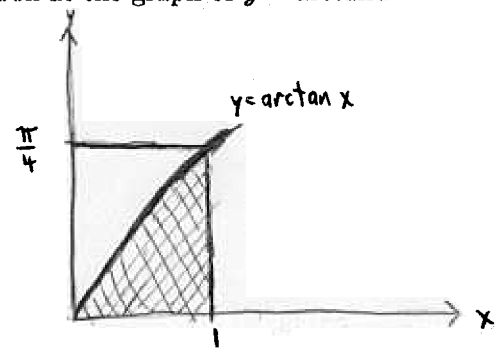
1 Integration Handout A

3. (a) $\arctan x$ is an odd function, so the areas on either side of the y-axis cancel out, i.e.,

$$\int_{-2}^{0} \arctan x \, dx = -\int_{0}^{2} \arctan x \, dx$$

Thus, $\int_{-2}^{2} \arctan x \, dx = 0$.

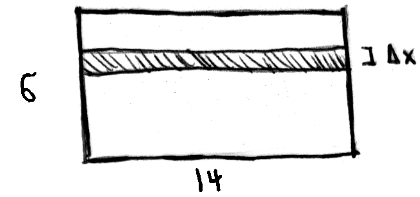
(b) Here we take a look at the graph of $y = \arctan x$:



Note that the sought-for area is difference between the area of the indicated rectangle and the area between $y = \arctan x$ and the y-axis. But this last area is given by an integral of the tan function with respect to y. That is:

$$\begin{aligned}
\int_0^1 \arctan x \, dx &= A_{rectangle} - A_{between\ curve\ and\ y-axis} \\
&= 1 \cdot \frac{\pi}{4} - \int_0^{\pi/4} \tan y \, dy \\
&= \frac{\pi}{4} - \ln|\sec y| \Big|_0^{\pi/4} \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

4. (a) We use the concept of the Riemann sum once again here. Divide the platter into n strips, making the divisions parallel to the long side (in other words, each strip is 14 inches long and $\Delta x = \frac{6}{n}$ inches wide):



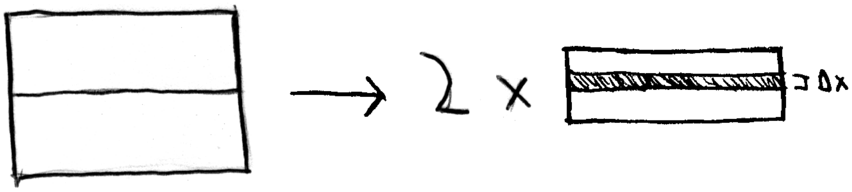
The mass of each of the strips is approximately equal to the area of the strip $(14 \cdot \frac{6}{n})$ times the density at a sample point in the strip. For convenience, the sample points (the x_i^*'s) I'll use here will be the distance from a particular long side of the platter to the farthest edge of the i^{th} strip; thus, $x_i^* = i\Delta x$. Then an approximation of the total mass is just the sum of the masses of the strips, or:

$$\sum_{i=1}^{n} 14 \cdot \rho(x_i^*)\Delta x = \sum_{i=1}^{n} 14 \cdot \rho\left(i \cdot \frac{6}{n}\right) \cdot \frac{6}{n}$$

(b) Since $\Delta x = \frac{6}{n}$, the requisite integral is limit of the sum in part (a) as n goes to infinity, or:

$$\int_0^6 14 \cdot \rho(x)\, dx$$

5. (a) This problem is very similar to #4: the only real change is that the ρ function is defined differently. The easiest way to modify the answer for #4 to accommodate this change is to split the platter right down the middle (length-wise) and calculate the cobalt mass for one of the halves; since the halves have the same mass, we simply double our calculation to get the mass of the cobalt on the entire platter.



Formally, the mass is:

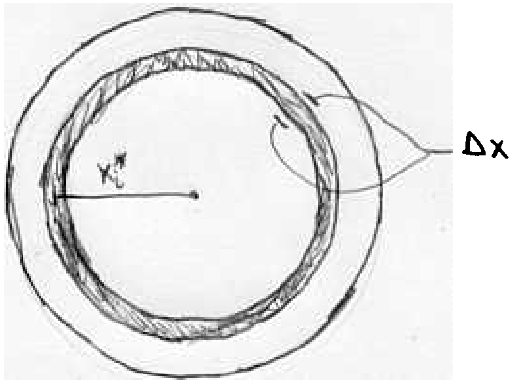
$$2\sum_{i=1}^{n} 14 \cdot \rho(x_i^*)\Delta x = 2\sum_{i=1}^{n} 14 \cdot \rho\left(i \cdot \frac{3}{n}\right) \cdot \frac{3}{n}$$

(Note that we have replaced $\frac{6}{n}$ with $\frac{3}{n}$ because the Riemann sum is only for half the platter, i.e., for 3 inches' worth.)

(b) Again, the total mass of cobalt is twice the cobalt mass on half of the plate.

$$2\int_0^3 14 \cdot \rho(x)\, dx$$

6. (a) This problem is different from #4 and #5 in that we are no longer dealing with rectangles. Nevertheless, the idea is the same: slice the area up into smaller areas whose densities can be well-approximated with a single value. Here, we slice up the circle into concentric rings of width $\Delta x = \frac{8}{n}$:



If x_i^* is the distance from the center of the circle to a point halfway between the inner and outer radii of the ith ring, then each ring has area of $\pi(x_i^* + \frac{\Delta x}{2})^2 - \pi(x_i^* - \frac{\Delta x}{2})^2$. This expression simplifies to $2\pi x_i^* \Delta x$ (see solution to problem #9 for the algebra). Finally, $x_i^* = \frac{2i-1}{2}\frac{8}{n}$ (which, incidentally, seems complicated, but does not figure into the actual calculation of the integral in part b). Then the mass of the ring is the area times $\rho(x_i^*)$. Again setting up the Riemann sum, the answer is:

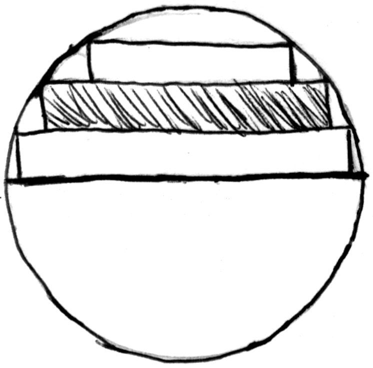
$$\sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x = \sum_{i=1}^{n} 2\pi \left((2i-1)\frac{4}{n}\right) \rho\left((2i-1)\frac{4}{n}\right) \Delta x$$

(b)

$$\int_0^8 2\pi x \rho(x)\, dx$$

Note that if $\rho(x) \equiv 1$, the integral turns into $\pi(8)^2$, the proper expression for the mass if the density is always 1.

7. Because $\rho(x)$ varies linearly and not radially, we cannot use the same approach as we could in #6. However, we can use rectangles as per #4 and #5:



Note that both the lengths and widths of the rectangles are changing now. Further note that again, $\rho(x)$ takes as its argument the distance from a line in the center of the plate, much like #5 did; thus, we can again split the plate into 2 equal halves, figure out the mass of cobalt on one of those halves, and double the answer at the end.

So consider this:

The width of each rectangle is $\Delta x = \frac{8}{n}$. The height of the ith rectangle (from the left) is $2\sqrt{8^2 - (x_i^*)^2}$, where $x_i^* = i\Delta x$ (can you see why?). Thus, the mass of cobalt on the entire plate should be:

$$2 \lim_{n\to\infty} \sum_{i=1}^{n} 2\sqrt{64 - (x_i^*)^2} \cdot \rho(x_i^*)\Delta x = 2\int_0^8 2\sqrt{64 - x^2} \cdot \rho(x)\, dx$$

9.

$$\begin{aligned} A_{kth\,ring} &= \pi\left(x_k^* + \frac{\Delta x}{2}\right)^2 - \pi\left(x_k^* - \frac{\Delta x}{2}\right)^2 \\ &= \pi\left((x_k^*)^2 + x_k^*\Delta x + \left(\frac{\Delta x}{2}\right)^2 - (x_k^*)^2 + x_k^*\Delta x - \left(\frac{\Delta x}{2}\right)^2\right) \\ &= 2\pi x_k^* \Delta x \end{aligned}$$