Problem Set #14

#2, 4, 7, 8 on Series Handout

§ 8.1 #39 / § 8.2 # 1, 2, 12, 16, 18, 20

2.) These are all geometric sums with ratio = $\frac{1}{3}$.

a) $\sum_{k=0}^{100} \left(\frac{1}{3}\right)^k$. $\quad a_0 = 1$, $n = 101$, $r = \frac{1}{3}$ $\Rightarrow$ $\sum_{k=0}^{100} \left(\frac{1}{3}\right)^k = \frac{1\left(1-\frac{1}{3}^{101}\right)}{1-\frac{1}{3}} = \boxed{\frac{3}{2}\left(1-\frac{1}{3}^{101}\right)}$

b) $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} = \boxed{\frac{3}{2}}$

$a_0 = 1$, $r = \frac{1}{3}$

c) $\sum_{k=2}^{100} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}\left(1-\frac{1}{3}^{99}\right)}{1-\frac{1}{3}} = \boxed{\frac{1}{6}\left[1-\frac{1}{3}^{99}\right]}$

$a_0 = \frac{1}{9}$

$n = 99$

$r = \frac{1}{3}$

d) $\sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \boxed{\frac{1}{6}}$

$a_0 = \frac{1}{9}$

$r = \frac{1}{3}$

4.) This is an infinite geometric sum with ratio x.

$$S = \lim_{n\to\infty} \frac{a_0(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{1}{1-x}\lim_{n\to\infty} x^n$$

Thus for S to be finite than $\lim_{n\to\infty} x^n$ has to be finite. If $|x| \geq 1$, then $\lim_{n\to\infty} x^n$ doesn't exist. Hence, $\sum_{k=0}^{\infty} x^k$ converges if $|x| < 1$. Converges to the function $\frac{1}{1-x}$.

7.) $f(x) = \sin(\pi x)$ would change her mind.

If n is an integer, then $\sin(\pi n) = 0$ for all n.

Thus, $\lim_{n \to \infty} \sin(\pi n) = 0$.

However, if x is a real number than

$\lim_{x \to \infty} \sin(\pi x)$ doesn't exist because the sine

function just oscillates between -1 and 1 as

x gets larger.

8.) $\sum_{k=1}^{\infty} \frac{1}{2^k + k} = \frac{1}{2+1} + \frac{1}{4+2} + \frac{1}{8+3} + \ldots + a_n + \ldots$

$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + b_n + \ldots$

But, $\frac{1}{2+1} < \frac{1}{2}$ and $\frac{1}{4+2} < \frac{1}{4}$ and $\frac{1}{8+3} < \frac{1}{8}$ ... and $a_n < b_n$.

Thus, $\sum_{k=1}^{\infty} \frac{1}{2^k + k} < \sum_{k=1}^{\infty} \frac{1}{2^k}$.

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges to a sum of S which is finite.

Than $\sum_{k=1}^{\infty} \frac{1}{2^k + k} < S$. Thus, $\sum_{k=1}^{\infty} \frac{1}{2^k + k}$ is finite. Thus,

it converges.

39. $a_n = \frac{1}{2n+3}$ is decreasing since $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

   (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is n convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

12. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

16. $\sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n \quad \Rightarrow \quad a = \frac{1}{e^2} = |r| < 1$, so the series converges to $\frac{1/e^2}{1 - 1/e^2} = \frac{1}{e^2 - 1}$.

18. $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

20. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^2 + 2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2 + 2n}\right) = 1 \neq 0.$$