PSET #12 - 6.7 #2,4,6 , 6.7 $\underline{\underline{EC}}$ #13 10/20/02

SOLUTIONS

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)\,dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)\,dt$.

4. (a) As in the preceding exercise, (1) $f(x) \geq 0$ and

(2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} f(x)\,dx = \frac{1}{2}(10)(0.2)$ [area of a triangle] $= 1$. So $f(x)$ is a probability density function.

(b) (i) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(0.1) = \frac{3}{20} = 0.15$

(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).

$P(X > 8) = \int_8^{10} f(x)\,dx = \frac{1}{2}(2)(0.1) = \frac{2}{20} = 0.10$. So $P(3 \leq X \leq 8) = 1 - 0.15 - 0.10 = 0.75$.

(c) We find equations of the lines from $(0,0)$ to $(6,0.2)$ and from $(6,0.2)$ to $(10,0)$, and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \leq x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \leq x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^6 x\left(\tfrac{1}{30}x\right)dx + \int_6^{10} x\left(-\tfrac{1}{20}x + \tfrac{1}{2}\right)dx = \left[\tfrac{1}{90}x^3\right]_0^6 + \left[-\tfrac{1}{60}x^3 + \tfrac{1}{4}x^2\right]_6^{10}$$
$$= \tfrac{216}{90} + \left(-\tfrac{1000}{60} + \tfrac{100}{4}\right) - \left(-\tfrac{216}{60} + \tfrac{36}{4}\right) = \tfrac{16}{3} = 5.\overline{3}$$

6. (a) $\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$

(i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t/1000}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{x} = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty}\int_m^x \frac{1}{1000}e^{-t/1000}\,dt = \frac{1}{2} \Rightarrow$

$\lim\limits_{x\to\infty}\left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln\frac{1}{2} \Rightarrow$

$m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

13. (a) First $p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^2 e^{-2r/a_0}\,dr$$

As in Exercise 12, we use (2) from that solution (with $b = -2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r\to\infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r\to\infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r\to\infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0$.

To find the maximum of p, we differentiate:

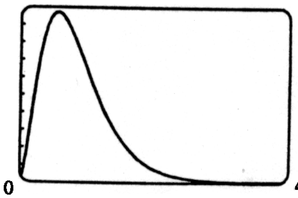
$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

$p'(r) = 0 \;\Leftrightarrow\; r = 0$ or $1 = \frac{r}{a_0} \;\Leftrightarrow\; r = a_0$ $[a_0 \approx 5.59 \times 10^{-11}\text{ m}]$. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9{,}684{,}098{,}979$$

With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.



(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds \;\Rightarrow\; P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds$. Using (2) from the solution to Exercise 12 (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64 + 16 + 2) - 1(2)\right]$$
$$= -\tfrac{1}{2}(82e^{-8} - 2) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} r p(r)\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0}\,dr$. Integrating by parts three times or using a CAS, we find that $\int x^3 e^{bx}\,dx = \frac{e^{bx}}{b^4}(b^3x^3 - 3b^2x^2 + 6bx - 6)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get $\mu = \frac{4}{a_0^3}\left[\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0$.