

Some Extra Credit Problems:

These problems explore alternatives to the remainder theorem for bounding the error between a function $f(x)$ and its n th Taylor polynomial $P_n(x)$ about a point a . In this development we assume that the Taylor series for e^x and $\sin x$ at $x = 0$ converge to their generating functions. (It is possible to demonstrate this without Taylor's Remainder but using differential equations. We will do this in the last unit of the course.)

1. Suppose we are interested in approximating $e = e^1$. Using the remainder theorem (applied to the expansion about $a = 0$), we get the bound $R_n(x) \leq \frac{M_{n+1}}{(n+1)!} |1 - 0|^{n+1}$, where M_{n+1} is a bound on the $n + 1$ st derivative of e^x on the interval $[0, 1]$. Indeed since every derivative of e^x is just e^x , the largest possible value is e . We get $R_n(x) \leq \frac{e}{(n+1)!}$.

On the other hand, assuming that we know that the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x . Then the error is obtained by taking the series starting with the $n + 1$ st term:

$$\begin{aligned} R_n(x) &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \\ &= \frac{1}{(n+1)!} \left[1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \dots \right] \\ &< \frac{1}{(n+1)!} \left[1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \dots \right] \end{aligned}$$

The expression inside the brackets is a geometric series with $r = \frac{1}{n+2} < 1$, hence convergent, with sum $\frac{1}{1 - \frac{1}{n+2}} \leq 3/2$ even when $n = 1$. Therefore we get $R_n(x) \leq \frac{3/2}{(n+1)!}$, which is in fact a better bound than we got with the remainder theorem.

2. Use the same argument to get that for any x , the error $R_n(x)$ between e^x and its n th Taylor polynomial at 0 is less than or equal to

$$\frac{|x|^{n+1}}{(n+1)!} \left[1 + \frac{x}{n+2} + \left(\frac{x}{n+2} \right)^2 + \dots \right]$$

If we choose n such that $|x| < n + 2$ (this is always true for sufficiently large n), then this geometric series converges. What is the bound that we get for $R_n(x)$?

3. By comparing the terms of the Taylor series expansion at $a = 0$ of the sine function to those of e^x , show that the error $R_n(x)$ for the sine function is less than or equal to the error $R_n(x)$ for the exponential function. Does the same hold for the cosine function?