APPENDIX F

L'Hôpital's Rule: Using Relative Rates of Change to Evaluate Limits

F.1 INDETERMINATE FORMS

In evaluating limits, we sometimes encounter situations in which our intuition might pull us in two different directions simultaneously. Many limits of this sort are called indeterminate forms; below are a few familiar examples.

i. $\lim_{x\to\infty} \frac{3x^2+1}{x^2}$	ii. $\lim_{h\to\infty} \frac{f(a+h)-f(a)}{h}$
iii. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$	iv. $\lim_{x\to 0^+} x \ln x$

We take these up one by one below.

i. $\lim_{x\to\infty} \frac{3x^2+1}{x^2}$
In the section on rational functions, we encountered limits like this. The numerator and the denominator both grow without bound. We will refer to this as a limit of the form $\frac{\infty}{\infty}$. Because the numerator is growing without bound we might expect the fraction to grow without bound; on the other hand, because the denominator is growing without bound we might expect the limit to be zero. We learned that in general the answer to $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ depends on the degree of the polynomials involved; the limit could be zero (degree of numerator < degree of denominator), a nonzero finite number (degrees equal), or unbounded (degree of numerator > degree of denominator). In the case of $\lim_{x\to\infty} \frac{3x^2+1}{x^2}$, we know that in fact the limit is 3. The answer depends on the *relative rates* at which the numerator and the denominator are growing.

ii. $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
Every time that we use the limit definition of the derivative we also have intuition pulling us in two directions. In evaluating $f'(a) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, both the numerator and the denominator are approaching zero. The former fact might make us think that the limit should be zero, while the latter could seem to imply that the

expression should be unbounded. We will refer to this as a limit of the form $\frac{0}{0}$. In fact, we know from our experience evaluating derivatives that $f'(a)$ might be any real number at all. Actually the value of $f'(a)$ depends on the *relative rates* at which Δy and Δx are approaching zero; the derivative is the ratio of these rates. We can evaluate the limit by comparing their rates of growth.

iii. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
In Chapter 15 we worked on this limit. In trying to evaluate it, we noticed how our intuition pulled us in two opposite directions. On the one hand, the base is approaching 1, and we know that 1 raised to any power is still 1, making us think that the limit should be 1. But, at the same time, the exponent is growing without bound, leading us to think that $\left(1+\frac{1}{n}\right)^n$ should grow without bounded as well, as $\lim_{n\to\infty} A^n$ is infinite if A is a constant greater than 1. As it turned out, the answer was somewhere in between; $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\ldots$.

iv. $\lim_{x\to 0^+} x\ln x$
This limit is of the form $0\cdot\infty$. The two factors, $(-\infty)$ and 0 pull us in different directions.

Each of the examples above is an indeterminate form. As you probably remember, we put a fair amount of work into evaluating the limits listed above. Will we need to do so much work to evaluate all limits like this? Fortunately, the answer is no; there is a quick and simple method for evaluating many indeterminate forms. This method, called l'Hôpital's Rule, depends heavily on the work we have already done in learning how to compute the derivatives of many different types of functions; it would not be possible to use this method had we not done that previous work.

Indeterminate Forms of Type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

The expression $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. The expression $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$ if $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$. These statements also hold if a is replaced by ∞, $-\infty$, or a one-sided limit.

L'Hôpital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

if the latter limit exists. This result also holds if a is replaced by ∞, $-\infty$, or a one-sided limit.

The proof of a special case of this rule is given on page 1114, but it is not very enlightening in terms of showing why the rule should work; so, before we give this proof, we will look at a more illuminating (but not rigorous) "explanation" of the rule. First, though, we'll get our feet wet by showing how l'Hôpital's Rule is used in the following examples.

◆ **EXAMPLE F.1** Compute $\lim_{x\to\infty} \frac{e^x}{x}$.

SOLUTION $\lim_{x\to\infty} \frac{e^x}{x}$ is of the form $\frac{\infty}{\infty}$, so l'Hôpital's Rule tells us

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

This reconfirms a fact we've asserted: e^x grows considerably faster than does x. ◆

◆ **EXAMPLE F.2** Compute $\lim_{x\to\infty} \frac{e^x}{x^3}$.

SOLUTION $\lim_{x\to\infty} \frac{e^x}{x}$ is of the form $\frac{\infty}{\infty}$, so l'Hôpital's Rule tells us

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2}$$

if the latter exists. But $\lim_{x\to\infty} \frac{e^x}{3x^2}$ is of the form $\frac{\infty}{\infty}$, so l'Hôpital's Rule can be applied again.

$$\lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x}$$

Here we can either apply l'Hôpital's Rule again or use the result of Example F.1: $\lim_{x\to\infty} \frac{e^x}{6x} = \infty$. ◆

In the next example we compare the growth rates of exponentials as compared with polynomials.

◆ **EXAMPLE F.3** Evaluate $\lim_{x\to\infty} \frac{P(x)}{b^x}$, where $P(x)$ is any polynomial of degree n and b is a constant greater than 1.

SOLUTION This limit is of the form $\frac{\infty}{\infty}$. When we apply l'Hôpital's Rule, we differentiate the numerator, obtaining a new polynomial of degree $n-1$, and we differentiate the denominator, obtaining $(\ln b)b^x$. If the degree of the new polynomial is 1 or greater, then as x tends toward infinity this polynomial will still tend to (positive or negative) infinity; the denominator will also still tend to infinity because it contains the b^x term. Thus, we will need to apply l'Hôpital's Rule again. But no matter how many times we differentiate the denominator, it will always be a constant times b^x and hence tend to infinity. On the other hand, once we have differentiated the numerator n times, it will be just a constant. So, we will be evaluating $\lim_{x\to\infty} \frac{K}{(\ln b)^n b^x} = 0$. ◆

The moral of Example F.3 is that, in the long run, exponentials dominate polynomials.

◆ **EXAMPLE F.4** Evaluate $\lim_{x\to 1} \frac{x-1}{\ln x}$.

SOLUTION This limit is of the form $\frac{0}{0}$. Before using l'Hôpital's Rule, we can look at this problem from a graphical perspective. The function $f(x) = x - 1$ and the function $g(x) = \ln x$ have equal slopes at $x = 1$. Because the rates at which $f(x)$ and $g(x)$ are changing are equal, we can conjecture that $\lim_{x\to 1} \frac{x-1}{\ln x}$ should be 1.

Numerical evidence also supports this, as choosing values of x near 1 gives values of $\frac{x-1}{\ln x}$ very nearly equal to 1. Now we use l'Hôpital's Rule to confirm our conjecture.

$$\lim_{x\to 1}\frac{x-1}{\ln x} = \lim_{x\to 1}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(\ln x)} \qquad \text{Apply l'Hôpital's Rule.}$$

$$= \frac{1|_{x=1}}{(1/x)|_{x=1}} \qquad \text{Compute the derivatives of the top and the bottom.}$$

$$= \frac{1}{1} \qquad \text{Evaluate them at } x = 1.$$

$$= 1$$

L'Hôpital's Rule provides us with a confirmation of our conjecture that this limit should be 1. ◆

"Explanation" (nonrigorous) of why l'Hôpital's Rule works in the special case that $f(a) = 0$ and $g(a) = 0$ and $f'(a)$ and $g'(a)$ exist; are finite, and $g'(a) \neq 0$.

If f is differentiable at $x = a$, then it can be approximated at $x = a$ by the line through the point $(a, f(a))$ with slope $f'(a)$.

$$y - f(a) = f'(a)(x-a) \text{ or } y = f(a) + f'(a)(x-a)$$

Similarly, near $x = a$, $g(x)$ is approximated by the line $y = g(a) + g'(a)(x - a)$.

$$\lim_{x\to a}\frac{f(x)}{g(x)} \approx \lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)} \qquad \text{Use the linear approximations to } f(x) \text{ and } g(x).$$

$$\approx \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)} \qquad f(a) \text{ and } g(a) \text{ are both zero by assumption.}$$

$$\approx \lim_{x\to a}\frac{f'(a)}{g'(a)} \qquad \text{Cancel the } (x-a) \text{ terms.}$$

$$\approx \frac{f'(a)}{g'(a)} \qquad \text{There are no } x\text{'s in the limit anymore.}$$

This result agrees with l'Hôpital's Rule as stated above. We see how the idea of local linearity means that it is reasonable that the rule should work; the value of the limit is the ratio of the rates of change of the numerator and denominator.

Proof of l'Hôpital's Rule for the Indeterminate Form $\frac{0}{0}$ in the special case that $f(a) = g(a) = 0$, f' and g' are continuous, $f'(a)$ and $g'(a)$ exist and are finite, and $g'(a)$ is nonzero.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-0}{g(x)-0}$$ Subtracting zero doesn't change anything.

$$= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$ $f(a)$ and $g(a)$ are zero by assumption.

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$ Multiply top and bottom by $\frac{1}{x-a}$.

$$= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$ Recognize that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$ and that $\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = g'(a)$.

The proof without the added assumptions is more difficult, and the proof for the form $\frac{\infty}{\infty}$ more difficult yet. We omit these proofs.

CAUTION

- L'Hôpital's Rule does *not* say to take the derivative of $\frac{f(x)}{g(x)}$; for that you would need to use the Quotient Rule. Instead, it says to take the derivatives of the numerator and denominator *individually* and compute the limit as x approaches a.
- L'Hôpital's Rule must only be applied when its conditions are met. In particular, if the limit does not have the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, l'Hôpital's Rule will generally give a wrong answer.

◆ **EXAMPLE F.5** Evaluate $\lim_{x\to 0}\frac{e^{3x}-1}{e^x-1}$.

SOLUTION This is of the indeterminate form $\frac{0}{0}$. The slope of $e^{3x}-1$ at $x=0$ is three times the slope of e^x-1 at $x=0$, so the value of the limit should be 3.

$$\lim_{x\to 0}\frac{e^x-1}{e^{3x}-1} = \lim_{x\to 0}\frac{\frac{d}{dx}(e^{3x}-1)}{\frac{d}{dx}(e^x-1)}$$ Apply l'Hôpital's Rule.

$$= \frac{3e^{3x}|_{x=0}}{e^x|_{x=0}}$$ Compute the derivatives individually.

$$= \frac{3}{1}$$ Evaluate.

$$= 3$$ ◆

◆ **EXAMPLE F.6** Evaluate $\lim_{x\to\infty}\frac{3x+1}{4x+17}$.

SOLUTION This is of the form $\frac{\infty}{\infty}$. Because this is a rational function where the degrees of the numerator and denominator are equal, it should have an asymptote at $y = 3/4$ (the ratio of the leading coefficients). We'll use l'Hôpital's Rule to confirm this.

$$\lim_{x\to\infty} \frac{3x+1}{4x+17} = \lim_{x\to\infty} \frac{3}{4}$$ Differentiate top and bottom and look at limit as x tends to infinity.

$$= \frac{3}{4}$$ Evaluate. ◆

◆ **EXAMPLE F.7** Evaluate $\lim_{x\to 0^+} \frac{\ln x}{1/x}$.

SOLUTION This is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$ Apply l'Hôpital's Rule.

$$= \lim_{x\to 0^+} -x$$ Simplify the fraction.

$$= 0$$ Evaluate. ◆

The Indeterminate Form $0 \cdot \infty$.

If one quantity is tending to zero and another is tending to infinity, then their product might be zero or infinity or anywhere in between. Consider the following examples. (Note that we are not using l'Hôpital's Rule, but rather just algebra to evaluate them.)

$\lim_{x\to\infty} \frac{1}{x^2} \cdot x$ is of the form $0 \cdot \infty$, but $\lim_{x\to\infty} \frac{1}{x^2} \cdot x = \lim_{x\to\infty} \frac{1}{x} = 0$.

$\lim_{x\to\infty} \frac{1}{x} \cdot x^2$ is of the form $0 \cdot \infty$, but $\lim_{x\to\infty} \frac{1}{x} \cdot x^2 = \lim_{x\to\infty} x = \infty$.

$\lim_{x\to\infty} \frac{39}{x} \cdot x$ is of the form $0 \cdot \infty$, but $\lim_{x\to\infty} \frac{39}{x} \cdot x = \lim_{x\to\infty} 39 = 39$.

Most limits that take the indeterminate form $0 \cdot \infty$ are harder to evaluate than the three just given. Fortunately, we can often convert the indeterminate form $0 \cdot \infty$ into the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ with a little algebraic manipulation.

◆ **EXAMPLE F.8** Evaluate $\lim_{x\to 0^+} x \ln x$.

SOLUTION $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \lim_{x\to 0^+} \frac{\ln x}{1/x}$. This latter limit is the one computed in Example F.7. The limit is 0. ◆

◆ **EXAMPLE F.9** Evaluate $\lim_{x\to\infty} xe^{-x}$. This is of the form $\infty \cdot 0$; we can convert it into the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$$ Convert $\infty \cdot 0$ into $\frac{\infty}{\infty}$.

$$= \lim_{x\to\infty} \frac{1}{e^x}$$ Apply l'Hôpital's Rule.

$$= 0$$ ◆

◆ **EXAMPLE F.10** Evaluate $\lim_{x\to 2}(x-2)^2 \ln(x-2)$. This is of the form $0 \cdot (-\infty)$; algebra converts it to the form $\frac{-\infty}{\infty}$.

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$$\lim_{x\to 2^+}(x-2)^2\ln(x-2) = \lim_{x\to 2^+}\frac{\ln(x-2)}{(x-2)^{-2}}$$ Convert $0\cdot(-\infty)$ into $\frac{\infty}{\infty}$.

$$= \lim_{x\to 2^+}\frac{\frac{1}{x-2}}{\frac{-2}{(x-2)^3}}$$ Apply l'Hôpital's Rule.

$$= \lim_{x\to 2^+}\frac{-(x-2)^2}{2}$$ Simplify.

$$= 0$$ Evaluate. ◆

The Indeterminate Forms 1^∞, ∞^0, and 0^0

In Chapter 15, when we were trying to evaluate the limit $L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ in which the base was tending to 1 and the exponent to infinity, we took the natural logarithm of each side, evaluated $\ln L$, and then used that to determine L itself. We can use the same technique in conjunction with l'Hôpital's Rule to evaluate other indeterminate forms of the types 1^∞, 0^0, or ∞^0.

Notation:

1^∞: We use the notation 1^∞ to denote the situation in which the base is tending toward 1 (not when the base *is* 1) and the exponent is increasing without bound. If the base actually is exactly 1, then the limit must be 1 no matter what the exponent is.

0^0: We use the notation 0^0 to denote the situation in which both the base and the exponent are approaching zero. This form is indeterminate because, on the one hand, 0 raised to any power is zero, but on the other hand, any nonzero base raised to the power of zero is one; intuition pulls us in two different directions.

∞^0: We use the notation ∞^0 to denote the situation in which the base is growing without bound while the exponent is approaching zero. This is an indeterminate form because any nonzero number raised to the power of zero is one, but a base growing unboundedly raised to a positive power should also be unbounded.

◆ **EXAMPLE F.11** Evaluate $\lim_{x\to 0^+} x^x$.

SOLUTION This is the indeterminate form 0^0. Assuming the limit exists, we let $L = \lim_{x\to 0^+} x^x$ and take natural logs.

$$L = \lim_{x\to 0^+} x^x$$

$$\ln L = \lim_{x\to 0^+} x\ln x$$ This is a new indeterminate form, $0\cdot(-\infty)$.

$$= \lim_{x\to 0^+}\frac{\ln x}{1/x}$$ Convert to form $\frac{-\infty}{\infty}$ so we can apply l'Hôpital's Rule.

$$= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$ Take derivatives of top and bottom.

$$= \lim_{x\to 0^+} -x$$ Simplify.

$$= 0$$

We are not done yet. We have determined that $\ln L = 0$, so L must equal 1. Thus, $\lim_{x\to 0^+} x^x = 1$. ◆

If you let $u = \frac{1}{x}$ then this example is the same as $\lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^u$ i.e. $\lim_{u \to \infty} \left(\frac{u+1}{u}\right)^u$

◆ **EXAMPLE F.12** Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$ (assuming the limit exists.)

SOLUTION This is of the form 1^∞, so we let $L = \lim_{x \to 0^+} (1+x)^{1/x}$ and take the natural log of each side.

$$L = \lim_{x \to 0^+} (1+x)^{1/x}$$

$$\ln L = \lim_{x \to 0^+} \frac{1}{x} \ln(1+x)$$ This is a new indeterminate form $0 \cdot (\infty)$.

$$\ln L = \lim_{x \to 0^+} \frac{\ln(1+x)}{x}$$ Convert to the form $\frac{0}{0}$.

$$\ln L = \lim_{x \to 0^+} \frac{1/(1+x)}{1}$$ Take derivatives according to l'Hôpital's Rule.

$$\ln L = \lim_{x \to 0^+} \frac{1}{1+x}$$ Simplify.

$$\ln L = 1$$

Because $\ln L = 1$, L must equal e. Therefore, $\lim_{x \to 0^+} (1+x)^{1/x} = e$. (Notice that this is the same as the $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ but with the change of variable $x = 1/n$.) ◆

PROBLEMS FOR APPENDIX F

For Problems 1 through 15, evaluate the following limits. (Note: *L'Hôpital's Rule is a fine tool, but it is not applicable to every problem.)*

1. $\lim_{x \to \infty} \frac{\ln x}{x}$

2. $\lim_{x \to 0^+} \frac{\ln x}{x}$

3. $\lim_{x \to \infty} \frac{x}{e^x}$

4. $\lim_{x \to 0} \frac{x}{e^{-x}}$

5. $\lim_{x \to \infty} \frac{100x^3}{e^x}$

6. $\lim_{x \to \infty} \frac{\ln(5+e^x)}{3x}$

7. (a) $\lim_{t \to 0} \frac{t^2+3}{2t^3+100t+1}$ (b) $\lim_{t \to \infty} \frac{t^2+3}{2t^3+100t+1}$

8. $\lim_{t \to 0} \frac{2t}{t^3}$

9. $\lim_{x \to \infty} 2x \cdot e^{-x}$

10. $\lim_{x \to -\infty} \frac{e^x+2}{x}$

11. $\lim_{x \to \infty} \frac{r^x}{x}$, where $0 < r < 1$.