Notes on Sharks and Food Fish for class on December 12, 2002

The predator-prey systems we've been studying are sometimes referred to as the Lotka-Volterra equations after the chemistt Lotka and the mathematician Volterra who used them in their work. Volterra's story was the one I talked about at the big morning meeting at the beginning of the course. I'll summarize it here; we're now in a position to understand the mathematics.

Fishing in the Adriatic Sea in the early 1900's was done by casting out large nets and hauling in fish. Fishermen then threw back inedible fish like sharks and kept the fish they wanted to sell at food markets. The Italian biologist Humberto D'Ancona noticed that during World War I, when fishing in the Adriatic sea was low due to the war effort, the percentage of food fish (hereafter referred to as 'fish') went down while the percentage of non-edible fish-eating fish (hereafter referred to as 'sharks') went up. (The effect was dramatic, with the percentage of sharks changing from 12% to 36% in some instances.) More vigorous fishing increased the percentage of food fish. He wondered why. He turned to his father-in-law, the mathematician Vito Volterra, who was able to answer the question using systems of differential eqatuions to model the predator/prey system of sharks and fish.

Let $S(t)$ be the number of hundreds of sharks in the Adriatic Sea at time t.
Let $F(t)$ be the number of hundreds of fish in the Adriatic Sea at time t.
Then the system of differential equations modelling the predator/prey interaction could be of the form

$$\begin{aligned}\frac{dS}{dt} &= -k_1S + k_2SF\\ \frac{dF}{dt} &= k_3F - k_4SF\end{aligned}$$

where k_1, k_2, k_3, and k_4 are all positive constants.
Alternatively, we could use a logistic model for fish and have

$$\begin{aligned}\frac{dS}{dt} &= -k_1S + k_2SF\\ \frac{dF}{dt} &= k_3F - k_4SF - k_5F^2\end{aligned}$$

where k_1, k_2, k_3, k_4, and k_5 are all positive constants.

With constants chosen appropriately, the trajectories in the FS-plane will be closed curves about the non-trivial equilibrium in the case of the former case and if k_5 is small enough will be spiraling in towards the non-trivial equilibrium point in the latter case.

Let's look at an example with concrete numbers (not chosen for their accuracy but rather for illustrative purposes). Suppose we have the following model:

$$\begin{aligned}\frac{dS}{dt} &= -0.4S + 0.1SF = .1S(-4+F)\\ \frac{dF}{dt} &= 0.3F - 0.3SF = .3F(1-S)\end{aligned}$$

This system of differential equations has equilibrium points at $(0,0)$ and $(4,1)$ – that is $F = 4$, $S = 1$, in the FS-plane. The solution trajectories for $F > 0$ and $S > 0$ are closed curves about $(4,1)$.

Let T denote the period of a cycle, i.e. the time it takes to get around a curve once. Then $F(0) = F(T)$ and $S(0) = S(T)$.

Claim 1: The average number of fish and sharks in a cycle are 4 hundred and 1 hundred respectively. (That is, the averages are the equilibrium levels.)

$$\text{Average number of fish} = \frac{1}{T}\int_0^T F(t)\,dt$$

We can't solve for F explicitly in terms of t so we'll use the differential equation to help us compute this. We'll calculate this as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= .1S(-4+F) \\
\frac{10}{S}\frac{dS}{dt} &= -4+F \\
F &= \frac{10}{S}\frac{dS}{dt}+4 \\
\int_0^T F\,dt &= \int_0^T \frac{10}{S}\frac{dS}{dt}\,dt + \int_0^T 4dt \\
\int_0^T F\,dt &= 10\ln S(t)|_0^T + 4t|_0^T \\
\int_0^T F\,dt &= 10(\ln S(T) - \ln S(0)) + 4T \\
\int_0^T F\,dt &= 4T \\
\frac{1}{T}\int_0^T F\,dt &= 4
\end{aligned}
$$

Similarly:

$$
\text{Average number of sharks} = \frac{1}{T}\int_0^T S(t)\,dt
$$

We can't solve for S explicitly in terms of t so we'll use the differential equation to help us compute this. We'll calculate this as follows:

$$
\begin{aligned}
\frac{dF}{dt} &= .3F(1-S) \\
\frac{10}{3F}\frac{dF}{dt} &= 1-S \\
S &= 1-\frac{10}{3F}\frac{dF}{dt} \\
\int_0^T S\,dt &= \int_0^T 1-\frac{10}{3F}\frac{dF}{dt}\,dt \\
\int_0^T S\,dt &= [1-10/3\ln F(t)]|_0^T \\
\int_0^T S\,dt &= T-10/3(\ln F(T)-\ln F(0)) \\
\int_0^T S\,dt &= T \\
\frac{1}{T}\int_0^T S\,dt &= 1
\end{aligned}
$$

Claim 1 has been proven. A more general statement can be proven similarly.
Claim 2: Fishing will raise the percentage of food fish and lower the percentage of sharks in a catch.

To model fishing in our example we can do the following:

$$
\begin{aligned}
\frac{dS}{dt} &= -0.4S+0.1SF-\lambda S = .1S(-4+F-\lambda) \\
\frac{dF}{dt} &= 0.3F-0.3SF-\lambda F = .3F(1-S-\lambda)
\end{aligned}
$$

Here λ varies with the intensity of fishing.

The new non-trivial equilibrium point for the system is $F = 4 + 10\lambda$ and $s = 1 - \frac{10}{3}\lambda$.

It follows from the proof of Claim 1 that our new averages for the cycles are these new levels, where the fish levels have gone up with λ and the shark levels have gone down with λ.