CHAPTER 15

# Take It to the Limit

## 15.1 AN INTERESTING LIMIT

In this chapter we begin with a concrete problem and after several iterations of the problem find ourselves hurtling toward a new and interesting perspective on something familiar. We'll begin with a brief look at inflation around the globe.

Many nations in the world suffer from runaway inflation. For instance, while the inflation rate in the United States in 1996 was about 3%, the inflation rate in Russia was about 22%. For Russia this was a considerable improvement over 1995 when its inflation rate was 131% and 1994 when its inflation rate was 2600%.[1] Brazil, Israel, and the former Yugoslavia have all had runaway inflation. For instance, Brazilians on a monthly salary habitually spend a large portion of their paychecks upon receipt because their buying power at the beginning of the month far exceeds that of the same sum at the end of the month. In the former Yugoslavia stores routinely used to shut down for a while during business hours to adjust prices in order to keep up with inflation! When inflation rates are high, interest rates on savings accounts become very high as well. In a country with runaway inflation it is not inconceivable to posit the existence of a bank offering 100% interest.

◆ **EXAMPLE 15.1** Let's suppose we put $10,000 in a bank account with a nominal interest rate of 100%, and we leave the money in the bank untouched for exactly one year. We will compute the amount it will grow to by the end of the year under various compounding schemes. Compounding interest more than once a year will allow interest to generate more interest, yielding an *effective* annual interest rate that is greater than the *nominal* interest rate of 100%.

(a) Suppose interest is compounded once a year. How much will be in the account by the end of the year?

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[1] Data from the *Boston Globe*, November 2, 1996.

(b) BankBoston has savings accounts for which interest is compounded monthly. Suppose the bank in this problem compounded monthly as well. How much money will we have by the end of the year?

(c) The Cambridge Savings Bank has savings accounts for which interest is compounded daily. If the bank in this problem compounds interest daily, how much money will be in the account by the end of the year?

(d) Suppose interest is compounded hourly. How much money will be in the account by the end of the year?

(e) What if interest is compounded every minute?

**SOLUTION** If $\$M_0$ is deposited in a bank account with a nominal interest rate of $r$ compounded $n$ times per year, then each compounding period the amount of money *increases* by $\frac{r}{n}$ times the amount of money present. In other words, each compounding period the money in the bank is multiplied by $(1 + \frac{r}{n})$.

For example, if the nominal interest is 8% per year ($r = 0.08$) and interest is compounded quarterly, then each quarter we will get $8\%/4 = 2\%$ interest. Each quarter the money in the account is multiplied by 1.02. In $t$ years there are $n \cdot t$ compounding periods. Therefore, after $t$ years the money grows to $M_0(1 + \frac{r}{n})^{nt}$. If $M_0 = 10{,}000$ and $r = 100\% = 1$, then after one year we have

$$\$10{,}000\left(1 + \frac{1}{n}\right)^n.$$

(a) Interest compounded once a year: $\$10{,}000(1 + 1)^1 = \$20{,}000$

(b) Interest compounded monthly: $\$10{,}000(1 + \frac{1}{12})^{12} \approx \$26{,}130.35$

(c) Interest compounded daily: $\$10{,}000(1 + \frac{1}{365})^{365} \approx \$27{,}145.67$

(d) Interest compounded hourly: There are 24 hours in a day, so there are $\left(24\ \frac{\text{hrs}}{\text{day}}\right)\left(365\ \frac{\text{days}}{\text{yr}}\right) = 8760$ hours in a year.

$$\$10{,}000\left(1 + \frac{1}{8760}\right)^{8760} \approx \$27{,}181.27$$

(e) Interest compounded every minute: There are 60 minutes in an hour, so there are $\left(60\ \frac{\text{min}}{\text{hr}}\right)\left(8760\ \frac{\text{hrs}}{\text{yr}}\right) = 525{,}600$ minutes per year.

$$\$10{,}000\left(1 + \frac{1}{525{,}600}\right)^{525{,}600} \approx \$27{,}182.80$$

Notice that at an interest rate of 100% there is a substantial difference between compounding the interest once a year and compounding it 12 times per year. In our example it amounts to over \$6000 for the year. On the other hand, the difference between compounding daily and compounding hourly amounts to less than \$36, and the difference between compounding every minute instead of every hour only increases the yield by less than two dollars. Suppose we let $n$ be the number of times we compound per year. If we let $n$ grow without bound, will the amount of money in the account after one year also grow without bound? It appears not; there seems to be a limit. ◆

Suppose $\$M_0$ is deposited in a bank account with a nominal interest rate of $r$ compounded $n$ times per year, and we take the limit as $n$ grows without bound. Then $M(t) = M_0 \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt}$, and we say that the account has an annual (nominal) interest rate of $r$ **compounded continuously**.

We can pose the question of how much money we would have after one year if we put \$10,000 in a bank with an annual interest rate of 100% compounded continuously. In other words, what is $\lim_{n\to\infty} 10{,}000\ (1 + \frac{1}{n})^n$? Evidence suggests that this quantity *may* be heading toward some finite number, but this remains to be shown, and the exact number remains to be identified. Take a guess on your own.

Before tackling $10{,}000 \lim_{n\to\infty}(1 + \frac{1}{n})^n$, let's take a brief look at some simpler limits.

**◆ EXAMPLE 15.2** Compute the following limits. In this problem, when we write $\lim_{n\to\infty}$ we mean $n$ increases through *the integers* without bound. (Compare Figures 15.1 and 15.2; in the latter the functions are defined for integers only.)

(a) $\lim_{n\to\infty} 2^n$ (b) $\lim_{n\to\infty} \left(1 + \frac{1}{10}\right)^n$ (c) $\lim_{n\to\infty} \left(1 - \frac{1}{10}\right)^n$

(d) $\lim_{n\to\infty} \left(1 + \frac{1}{100{,}000}\right)^n$ (e) $\lim_{n\to\infty} a^n$ for some constant $a > 0$

**SOLUTION**

| | |
|---|---|
| (a) $\lim_{n\to\infty} 2^n = \infty$ | Each time $n$ increases by 1 the expression doubles. |
| (b) $\lim_{n\to\infty}(1 + \frac{1}{10})^n = \infty$ | Each time $n$ increases by 1 the expression increases by 10%. (See Figure 15.2(i).) |
| (c) $\lim_{n\to\infty}(1 - \frac{1}{10})^n = 0$ | Each time $n$ increases by 1 the expression decreases by 10%.(See Figure 15.2(ii).) |
| (d) $\lim_{n\to\infty}(1 + \frac{1}{100{,}000})^n = \infty$ | $1 + \frac{1}{100{,}000}$ is larger than 1; as $n$ increases by 1 the expression increases by a positive percentage (0.001%) so the increase *increases* with $n$. $(1 + \frac{1}{100{,}000})^n$ is increasing at an increasing rate. |
| (e) $\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{for } a > 1 \\ 1 & \text{for } a = 1 \\ 0 & \text{for } 0 < a < 1 \end{cases}$ | If we know $\lim_{n\to\infty} b^n = \infty$ for $b > 1$ then it follows that $\lim_{n\to\infty} a^n = 0$ for $0 < a < 1$, since we can write $a^n$ as $(\frac{1}{b})^n = \frac{1}{b^n}$ for some $b > 1$. |

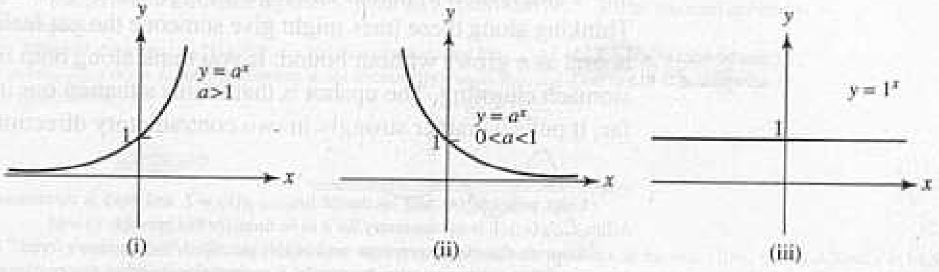

Figure 15.1

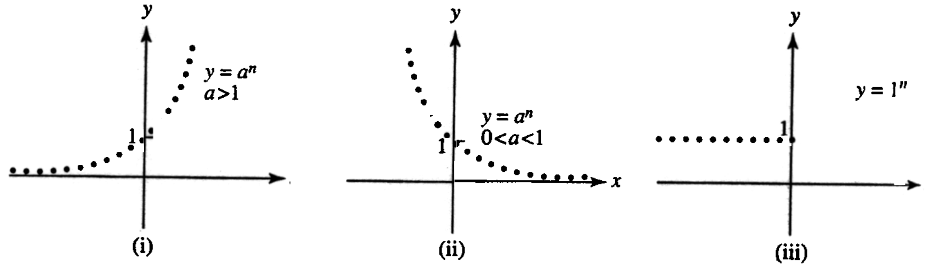

Figure 15.2 ◆

**EXERCISE 15.1** Convince yourself that none of these answers would change if we took the limit as increases without bound and $n$ is *not* restricted to integer values.

Let's look at another set of examples.

◆ **EXAMPLE 15.3** Compute the following limits.

$$\text{(a)}\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \qquad \text{(b)}\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10} \qquad \text{(c)}\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{100{,}000}$$

**SOLUTION** For $k$ any positive constant, $x^k$ is a continuous function. Therefore we can apply li principle (4) to get[2]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^k=\left[\lim_{n\to\infty}\left(\ +\frac{1}{n}\right)\right]^k=1^k.$$

We apply this logic to each of the limits in this example.

(a) $\lim_{n\to\infty}(1+\frac{1}{n})^2=1^2=1$

(b) $\lim_{n\to\infty}(1+\frac{1}{n})^{10}=1^{10}=1$

(c) $\lim_{n\to\infty}(1+\frac{1}{n})^{100{,}000}=1^{100{,}000}=1$ ◆

Then what is $\lim_{n\to\infty}(1+\frac{1}{n})^n$? Can our intuition get us anywhere? On one hand quantity $(1+\frac{1}{n})$ gets arbitrarily close to 1 as $n\to\infty$, and 1 raised to any power is 1 you might very well have the gut feeling that this limit will be 1. On the other hand any positive $n$ the quantity $(1+\frac{1}{n})$ is always bigger than 1; if we take any fixed numb $a>1$ and raise it to higher and higher powers, the resulting number grows without bou Thinking along these lines might give someone the gut feeling that $(1+\frac{1}{n})^n$ grows with bound as $n$ grows without bound. If you think along both lines simultaneously the result stomach churning. The upshot is that in this situation our intuition isn't going to get u far; it pulls us rather strongly in two contradictory directions. The situation is too subtl

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[2] Limit principle (4) tells us that if $\lim_{x\to a} g(x)=L$ and $h(x)$ is continuous at $x=L$, then $\lim_{x\to a} h(g($ $h(\lim_{x\to a} g(x))$. It is not necessary for $a$ to be finite for this principle to hold.

[3] Stomach-churning expressions such as this are called "indeterminate forms." In such cases our intuition pulls u two very different directions simultaneously. A computational method that enables us to deal efficiently with indetermin is introduced in Appendix F: l'Hôpital's Rule.

Let's return to Example 15.1. Recall that we have deposited \$10,000 into a bank with a nominal annual interest rate of 100% and left it for one year. If the interest were compounded $n$ times a year we would have \$10,000$(1 + \frac{1}{n})^n$. The question is whether $(1 + \frac{1}{n})^n$ increases without bound or approaches some limiting value. (When looked at in this context, it seems unreasonable that the limit would be 1.) Let's experiment by returning to the numerical approach suggested by Example 15.1 and evaluating $(1 + \frac{1}{n})^n$ for large $n$. The largest value we looked at in Example 15.1 was $n = 525{,}600$. Below are the results of evaluating $(1 + \frac{1}{n})^n$ for various values of $n$ using a TI-83. All the digits displayed by this calculator are recorded here.

For $n = 525{,}600$, the TI-83 gives 2.718279215.

For $n = 1{,}000{,}000$, the TI-83 gives 2.718280469.

For $n = 10^{10}$, the TI-83 gives 2.718281828.

For $n = 10^{15}$, the TI-83 gives 1.

What are we to make of this? For starters, look at the very last result. Do you honestly think that $(1 + \frac{1}{10^{15}})^{10^{15}} = 1$? No. The result *must* be larger than 1. As $n$ increases, $(1 + \frac{1}{n})^n$ increases; we know this from the context of the problem. What in fact is happening is that the calculator has treated $(1 + \frac{1}{10^{15}})$ as 1 and then computed $1^{10^{15}}$ and arrived at 1. Therefore, when considering the numerical results from the calculator, we need to disregard this one. If we evaluate $(1 + \frac{1}{n})^n$ for $n$ larger than $10^{10}$, the TI-83 will keep giving us 2.718281828 until $n$ gets so large that the TI-83 throws up its little calculator hands and gives us the number 1. The results of our numerical investigations might lead you to wonder whether the number 2.718281828 has some significance. Where have you seen this number before? If you make your calculator display (to the best of its ability) the number $e$, it will match up, decimal for decimal, with 2.718281828.[4] This might lead you to conjecture that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

## What Happens to the Limit as $n$ Grows Without Bound? Does the Limit Equal $e$?

Looking at numerical data and conjecturing that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$ is great, but if we want to verify this conjecture[5] we cannot simply try to match up decimal places. There are two logical difficulties. First of all, $e$ is an irrational number; $e$ cannot be written as a finite decimal or as an infinite repeating one, so we cannot ever hope to match all the decimal places for $e$. Second, and even more fundamental, is the question of where all of these decimal places for $e$ are coming from to begin with. We have simply **defined** $e$ to be the number "$a$" such that the derivative of the function $a^x$ is $a^x$.[6] We have to rely exclusively on

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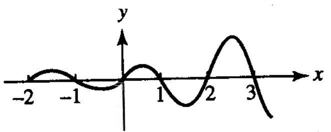
[4] If you get hold of a few more digits for $e$, you'll get 2.718281828459 . . . . $e$ is an irrational number; its decimal expansion is nonrepeating.

[5] We will show $\lim_{x\to\infty}(1 + \frac{1}{x})^x = e$ and then conclude that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$. We must be careful. If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} f(n) = L$, but the converse is not necessarily true. If $\lim_{n\to\infty} f(n) = L$ it is possible that $\lim_{x\to\infty} f(x) \neq L$. See the figure below.



[6] The Swiss mathematician Leonhard Euler, who first introduced the number $e$ in the mid-1700s, in fact *defined* $e$ to be the limit in question (although a rigorous foundation for limits was not laid down until the 1800s). In this text we have not followed the actual historical development and must now show that $e$ as we defined it is the same $e$ that Euler defined.

this definition of $e$. While the calculator has been instrumental in suggesting the conjecture, it has no role in actually proving it.[7]

Knowing that the derivative of $e^x$ is $e^x$, we concluded, after some work, that the derivative of its inverse function $\ln x$ is $1/x$; this information will turn out to be useful in verifying our conjecture. If we can show $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ we can conclude that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

## Is $\lim_{x\to\infty}(1+\frac{1}{x})^x$ Really $e$?

Let's use the longstanding tactic of naming what we are looking for. (What follows is valid if we assume this limit exists and is finite.)

$$\text{Let } B = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

We have a variable in the exponent; we'd like to "bring it down" in order to make the expression on the right more tractable, so we'll take the natural logarithm of both sides of the equation:

$$\ln B = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

Because the logarithm is a continuous function, this is equivalent to writing[8]

$$\ln B = \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to\infty} x\,\ln\left(1+\frac{1}{x}\right)$$

As $x \to \infty$ we have the product of two numbers, one of which is going toward zero (because $\ln 1 = 0$), and the other which is growing without bound.[9] $\ln(1+\frac{1}{x})$ is approaching $\ln 1 = 0$, but it is being multiplied by $x$, which is approaching $\infty$. It is still hard to decipher what is going on. Our area of expertise is more along the lines of taking the limit as something tends toward zero, and this may tie in with our defining characteristic of $e$. Let's try a substitution in hopes of figuring out this limit. Substitution is a tool for transforming the unfamiliar into the familiar.

$$\text{Let } h = \frac{1}{x}. \quad \text{As } x \to \infty, h \to 0.$$

With this substitution, $\lim_{x\to\infty} x\ln(1+\frac{1}{x})$ becomes $\lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\frac{\ln(1+h)}{h}$

$$\ln B = \lim_{h\to 0}\frac{\ln(1+h)}{h}$$

This latter limit is easier to evaluate. In fact, you may recognize that it is the definition of the derivative of $\ln x$ at $x = 1$. Verify, using Figure 15.3, that the slope of the secant line

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[7] At this point, we don't know how the calculator has arrived at its decimal expansion of $e$. Numerically all *we* have shown that $e$ lies between 2.7 and 2.8. If we can prove our conjecture, then we have a way of numerically approximating $e$.

[8] If $f$ is a continuous function, then $f(\lim_{x\to a} g(x)) = \lim_{x\to a} f(g(x))$. This is limit principle (4).

[9] Here's another "indeterminate form": $\infty \cdot 0$. On the one hand, any finite number multiplied by zero is zero. On the other hand, if a number is growing without bound and we multiply it by any positive number, the product ought to grow without bound. (Of course, the second factor is not identically zero; it is just tending toward zero. Neither is the first factor "$\infty$"; it is just growing without bound.) Again, our gut reaction to the problem pulls us in two different directions and the result churns the stomach

through $(1, 0)$ and $(1+h, \ln(1+h))$ is

$$\frac{\ln(1+h)-\ln 1}{(1+h)-1} = \frac{\ln(1+h)}{h}.$$

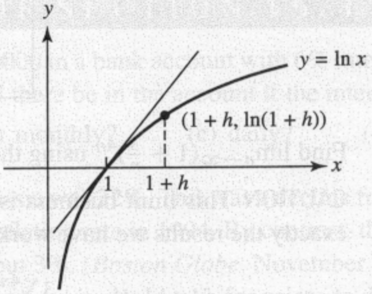

Figure 15.3

$$\text{So } \lim_{h\to 0}\frac{\ln(1+h)}{h} = \frac{d}{dx}(\ln x)\Big|_{x=1}.$$

The derivative of $\ln x$ is $\frac{1}{x}$, and evaluating at $x = 1$ gives 1. Therefore, $\ln B = 1$. We were looking for $B$, not $\ln B$; exponentiating shows that $B = e^1 = e$. Eureka!

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$

Not only have we gained a whole new perspective on $e$, but we now have a method of approximating $e$ numerically.

Having computed $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$, we can compute variations on this limit.

### ◆ EXAMPLE 15.4

Show that for any constant $r$, $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$.

**SOLUTION** To evaluate this limit we will use the technique employed so successfully above; we will rename our variables. (Changing variables is a standard mathematical technique for converting something that looks a bit unfamiliar to something well known and having familiar structure.) We would like to replace $\frac{r}{n}$ by $\frac{1}{m}$ so we can utilize our previous result. If $\frac{1}{m} = \frac{r}{n}$, then $mr = n$, so $m = \frac{n}{r}$. Therefore, we use the substitution $m = \frac{n}{r}$. We can assume that $r \neq 0$, because the case $r = 0$ can be easily handled independently.

$$\text{As } n \to \infty, \quad m = \frac{n}{r} \to \infty \text{ as well, because } r \text{ is a constant.}$$

Thus,

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mr} = \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^r.$$

But $r$ is a constant, so this is equivalent to[10]

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[10] We're using limit principle (4) again. If $f$ is continuous, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$.

$$\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{r}=e^{r}.$$

We conclude that

$$\boxed{\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}=e^{r}.}$$ ◆

◆ **EXAMPLE 15.5** Find $\lim_{n\to\infty}(1+\frac{3}{n})^{4n}$ using the result of Example 15.4.

**SOLUTION** CAUTION This limit business is subtle. We don't want to ad lib. Instead, we need to use exactly the results we have worked so hard to get.

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n}=\lim_{n\to\infty}\left[\left(1+\frac{3}{n}\right)^{n}\right]^{4}=[e^{3}]^{4}=e^{12}$$ ◆

### Implications of the Fact that $\lim_{n\to\infty}(1+\frac{r}{n})^{n}=e^{r}$

*Recall:* If you put $\$M_0$ in a bank account with nominal annual interest rate $r$ compounded $n$ times per year, then $M(t)$, the amount of money in the account after $t$ years, is given by

$$\boxed{M(t)=M_0\left(1+\frac{r}{n}\right)^{nt}.}$$

If we let the number of compounding periods increase without bound, we obtain

$$M(t)=M_0\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}=M_0\lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{n}\right]^{t}.$$

Having shown that $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}=e^{r}$, we obtain $M(t)=M_0[e^{r}]^{t}=M_0e^{rt}$. Therefore, if a bank with nominal annual interest rate $r$ compounds interest continuously,[11] then the money grows according to

$$\boxed{M(t)=M_0e^{rt}.}$$

**EXERCISE 15.2** You plan to deposit a fixed sum of money into one of two bank accounts and leave it there for several years. Which is a better choice of accounts, an account with a nominal interest

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[11] Do banks really do this? Certainly some banks compute interest on savings accounts every day. If the nominal annual inte rate is 5% and interest is compounded daily, then

$$M(t)=M_0\left(1+\frac{0.05}{365}\right)^{365t}\approx M_0(1.051267)^{t}.$$

If we modeled this situation using continuous compounding, we would have

$$M(t)=M_0e^{0.05t}\approx M_0(1.051271)^{t}.$$

Depending on the situation, the latter model may be quite reasonable and simpler to use. (Even using the equation $M(t)$ $M_0\left(1+\frac{0.05}{365}\right)^{365t}$ distorts reality a bit, because in reality if interest is compounded at the end of each day, $M(t)$ ought to a step function with a point of discontinuity each time interest is computed, and yet this function is continuous.)