

Solutions to the Series Exam: Exam 2

November 25, 2002: Math 1b

1. (a) The series converges, so the terms of the series must be going towards zero. (If they were not, the series would diverge by the Nth Term Test for Divergence.) Therefore the answer is $\lim_{n \rightarrow \infty} a_n = 0$.
(b) To say that the series converges to e is, by definition, to say that the sequence of partial sums tends to e . b_n is the n th partial sum, so $\lim_{n \rightarrow \infty} b_n = e$.
Note that while e is approximately 2.7, it is not 2.7, and the limit is unique, so the limit is not 2.7.
2. (a) $a_0 = f(2)$ and $f(2)$ is negative, so a_0 is negative.
(b) $a_1 = f'(2)$ and f is a differentiable function with a local min at $x = 2$, so f has a horizontal tangent line at $x = 2$. Consequently both the first derivative at $x = 2$ and a_1 are zero.
(c) $a_2 = \frac{f''(2)}{2}$ and f is concave up at $x = 2$, so $f''(2) > 0$. Consequently a_2 is positive.
3. (a) The series is centered at $x = 7$, so the interval of convergence (give or take its endpoints) will be centered about $x = 7$. The series converges at $x = -1$, which is 8 units away from 7. Therefore, the radius of convergence must be at least 8 and the series is guaranteed to converge for $[7 - 8, 7 + 8)$, or $[-1, 15)$. The reason for putting square brackets around the -1 is that we are told that the series converges there.
(b) This series converges. It is obtained by evaluating the series at $x = 9$ and $x = 9$ is firmly within the interval of convergence given in (a).
4. (a) In this case, S_n is a strictly decreasing sequence. (monotone decreasing). Recall that if a sequence is increasing and bounded then it converges. Similarly, if a sequence is decreasing and bounded, it converges. (It is bounded above by zero and below by m .) Thus by definition, the series converges.
(b) There is not enough information here. On the one hand, if all the a'_k 's are zero, the condition given is satisfied and the series converges. On the other hand, if $a_k = (-1)^k$ the series still satisfies the condition given but the series diverges.
(c) There is not enough information here. If we are told that all the terms in question are positive, then convergence is granted by the direct comparison test. However, we could take $b_k = 0$ and $a_k = -1$ for all k which would satisfy the conditions given while the series $\sum a_k$ clearly diverges.

5. We are asked to determine the convergence or divergence of the following infinite series:

a) $\sum_{n=1}^{\infty} \frac{n^4}{(n^3-10)^2}$

The denominator is very much like n^6 , so our intuition is that the entire series is very much like the series $\sum n^4/n^6 = \sum 1/n^2$. Since this latter series is convergent (a p -series with $p = 2 > 1$) we think that our series will be convergent. However we must justify this opinion by appealing to some convergence tests. We may try a direct comparison, but unfortunately $(n^3 - 10)^2 < n^6$, so that $\frac{1}{(n^3-10)^2} > \frac{1}{n^6}$ and in all the terms of our series are *greater* than the terms of $\sum 1/n^2$. So this kind of direct comparison doesn't work. This is really a job for the limit comparison test: since $\lim_{n \rightarrow \infty} \frac{\frac{n^4}{(n^3-10)^2}}{\frac{1}{n^2}} = 1$, the limit comparison test says that these two series converge or diverge together, so that they both converge. If we really like the direct comparison test, we can finagle it to make it work: for sufficiently large n , $\frac{n^4}{(n^3-10)^2} > \frac{1}{2n^2}$ or $\frac{1}{n^{3/2}}$.

b) $\sum_{n=1}^{\infty} (-1)^n \frac{5n^2-1}{17n^2+3n}$. Notice that the limit of the absolute value of the general term is $\lim |(-1)^n \frac{5n^2-1}{17n^2+3n}| = 5/17$, which is not zero. Therefore the general term does not converge to zero (a sequence converges to zero if and only if it converges to zero in absolute value), so by the divergence test (aka the n th term test) the series diverges. Some people simply removed the $(-1)^n$ factor from the general term of the series: it *is* true that the $(-1)^n$ can be omitted when testing to see whether the general term goes to zero, but you should give some indication or comment that this is what you are doing. Those who said, "The series diverges by the Alternating Series Test" lost a point, since the Alternating Series Test does not show the divergence of a series.

c) $\sum_{n=1}^{\infty} \frac{4 \cos n}{3^n}$. We will show that the series is absolutely convergent, hence convergent. This means that we will show that the series $\sum_{n=1}^{\infty} \frac{|4 \cos n|}{3^n}$ converges. Since now the terms are non-negative, we can make the direct comparison $4|\cos n| \leq 4$ and compare the series to four times the convergent geometric series $\sum_{n=1}^{\infty} (1/3)^n$ ($|r| = 1/3 < 1$), so the series is absolutely convergent.

Comments: i) It is not correct to just say $\cos n \leq 1$ and proceed by direct comparison, because comparison is only valid for series with non-negative terms, and $\cos n$ is sometimes negative – I took off two points for this. It is better to say $|\cos n| \leq 1$ and then do the comparison, but again, you were asked to say what you were doing. I did not give full credit unless I saw some awareness of the fact that absolute convergence implies convergence.

ii) There were some other mistakes that many students made. A common one was to say that the $\cos n$ makes the series alternating and to then try to apply the Alternating Series Test. Alternation means that the signs go $+, -, +, -, \dots$. In contrast, the signs of $\cos n$ for $n = 1, 2, 3, \dots$ go

+, -, -, -, +, +, +, -, -, -, +, +, +, +, ... i.e. they change every 3.14159... terms – this is not alternation! Even more seriously, if we take the absolute value, the terms are not decreasing due to the oscillating nature of the cosine. On a related note, many people tried to apply the ratio test, and they claimed that the quantity $|\cos(n+1)/\cos(n)|$ approached 1. Just because the numerator and denominator of a fraction are at most 1 does not mean that the fraction itself is at most 1 – e.g. $\frac{1/n^2}{1/2^n}$ goes to infinity and $\frac{1/2^n}{1/n^2}$ goes to 0. In fact this ratio of values of the cosine is very nasty: one can show that it takes arbitrarily large values as well as values arbitrarily close to zero (and everything in between).

d) $\sum_{n=10}^{\infty} \frac{(-1)^{n+1} 10}{\sqrt{\ln n}}$.

This really is an alternating series; the terms are decreasing in absolute value (since $\ln n$ is increasing) and approach zero. Therefore the alternating series test does (at last) apply to show the convergence of this series.

6. (a) Let a_n be $\frac{(x-3)^k}{5^k(k+1)}$, the n -th term of the series. To use ratio test to find the radius of convergence, we look at $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, which is

$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{k+1}/5^{k+1}(k+1+1)}{(x-3)^k/5^k(k+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)(k+1)}{5(k+2)} \right|$$

Since $\lim_{n \rightarrow \infty} \frac{k+1}{k+2} = 1$, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x-3}{5} \right|$. By ratio test, the series converges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x-3}{5} \right| < 1$, and diverges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x-3}{5} \right| > 1$. So the endpoints of the interval of convergence are the points where $\left| \frac{x-3}{5} \right|$ exactly equals 1, i.e. the points where $|x-3| = 5$. Hence the radius of convergence is 5.

- (b) Let a_n be $\frac{nx^n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$, the n -th term of the series. Again we look at $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, which is

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}/1 \cdot 3 \cdot 5 \cdot \dots \cdot [2(n+1)-1]}{nx^n/1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} \cdot \frac{(n+1)x}{n} \right|. \end{aligned}$$

Note that the notation $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$ means the product of all *odd* numbers between 1 and $2n-1$, and

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} = \frac{1}{2n+1}.$$

Hence

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{(2n+1)n} \right| = 0.$$

(Because $\lim_{n \rightarrow \infty} \frac{n+1}{(2n+1)^n} = \lim_{n \rightarrow \infty} \frac{1}{4n+1} = 0$ by L'Hopital's rule.)
 Thus $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ for all x , and by ratio test, the series converges for all x . This means the radius of convergence is ∞ .

7. (a) Taylor series for $\frac{1}{1-u}$ around 0 is

$$\frac{1}{1-u} = \sum_{i=0}^{\infty} u^i.$$

Let's do substitution $u = 4x^2$. We'll get

$$\frac{1}{1-4x^2} = \sum_{i=0}^{\infty} (4x^2)^i = \sum_{i=0}^{\infty} 4^i x^{2i}.$$

Multiplying by x^3 we'll get

$$\frac{x^3}{1-4x^2} = x^3 \sum_{i=0}^{\infty} 4^i x^{2i} = \sum_{i=0}^{\infty} 4^i x^{2i+3}.$$

(b) Let's find the radius of convergence first. In order to do this we'll use ratio test:

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i \rightarrow \infty} \left| \frac{4^{i+1} x^{2(i+1)+3}}{4^i x^{2i+3}} \right| = \lim_{i \rightarrow \infty} 4x^2 = 4x^2.$$

So, our series will converge for $4x^2 < 1$ or $x^2 < \frac{1}{4}$ or $-\frac{1}{2} < x < \frac{1}{2}$.
 Hence the radius of convergence is $R = \frac{1}{2}$.

Now let us check whether the series is convergent at the end points $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

$x = -\frac{1}{2}$ In this case the series will look like

$$\sum_{i=0}^{\infty} 4^i \left(-\frac{1}{2}\right)^{2i+3} = \sum_{i=0}^{\infty} \frac{(-1)^{2i+3} 4^i}{2^{2i+3}} = - \sum_{i=0}^{\infty} \frac{4^i}{4^i \cdot 2^3} = - \sum_{i=0}^{\infty} \frac{1}{8}.$$

This series is divergent by N -th term test.

$x = \frac{1}{2}$ In this case we'll have

$$\sum_{i=0}^{\infty} 4^i \left(\frac{1}{2}\right)^{2i+3} = \sum_{i=0}^{\infty} \frac{4^i}{2^{2i+3}} = \sum_{i=0}^{\infty} \frac{4^i}{4^i \cdot 2^3} = \sum_{i=0}^{\infty} \frac{1}{8}.$$

This series is also divergent by N -th term test.

Finally, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

(c)

$$\begin{aligned}\int \frac{x^3}{1-4x^2} dx &= \int \left(\sum_{i=0}^{\infty} 4^i x^{2i+3} \right) dx = \\ &= \sum_{i=0}^{\infty} \left(\int 4^i x^{2i+3} dx \right) = C + \sum_{i=0}^{\infty} \frac{4^i x^{2i+4}}{2i+4}.\end{aligned}$$

8. A ball is dropped from a height of Y meters. Each time it hits the ground it rebounds to a height that is 40% of the height from which it just fell. It bounces ad infinitum. The ball travels a total of 13 meters. From what height was it originally dropped?

Solution: The distance travelled is $Y + 2\frac{4}{10}Y + 2\left(\frac{4}{10}\right)^2 Y + 2\left(\frac{4}{10}\right)^3 Y + \dots$ (We count twice starting at the second term because the ball travels upwards and downwards after the first downward travel of Y meters). Therefore

$$13 = Y + 2 \sum_{n=1}^{\infty} \left(\frac{4}{10} \right)^n$$

The sum is a geometric series with $a = r = \frac{4}{10}$, and so $\sum_{n=1}^{\infty} \left(\frac{4}{10} \right)^n = \frac{\frac{4}{10}}{1 - \frac{4}{10}} = \frac{2}{3}$. Therefore $13 = Y + 2\frac{2}{3}Y = \frac{7}{3}Y$ and $Y = 7$.

- 9 (a) Plugging $x = 1.1$ into the Taylor series for $\ln x$ about $x = 1$ yields:

$$\ln 1.1 = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \frac{0.1^5}{5} - \dots$$

This is an alternating series, whose terms are decreasing in magnitude and tending towards zero. Therefore, we can apply the alternating series error estimate which says that the error made by approximating $\ln 1.1$ by the first three terms is less than the next term not taken into the sum, that is

$$|\text{error}| < \frac{0.1^4}{4}.$$

The third partial sum is an overestimate, as one sees by plotting the first few partial sums—they zigzag around the actual sum of the series, with the first one being larger, the second smaller, the third larger... than the actual sum.

- (b) The series giving $\ln 0.5$ is no longer alternating, so the Taylor remainder theorem must be used. Since we are approximating $\ln 0.5$ by the third Taylor polynomial centered at $a = 1$ and evaluated at $x = .5$, the absolute value of the error is given by:

$$|\text{error}| = \frac{|f^{(4)}(c)|(.5 - 1)^4}{4!},$$

where $.5 \leq c \leq 1$ and $f(x) = \ln x$.

The fourth derivative of $\ln(x)$ is $-\frac{6}{x^4}$. Therefore

$$|f^{(4)}(c)| = \frac{6}{c^4} \leq \frac{6}{0.5^4} = 96$$

for $0.5 \leq c \leq 1$,

and a bound on the error is $96 \times .5^4/4! = 1/4$.

The approximation is larger than the actual sum, since all terms of the series are negative.

10. Let $f(x) = \sin(x)$. Then we compute

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f''\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f'''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

We then obtain

$$T_3(x) = \sum_{n=0}^{n=3} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{4} \left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{6}\right)^3$$