Solutions to the First Exam for Math 1b: October 24, 2002

1. Problem 1.

   (a) There were two correct ways to do this problem: one was via cylinders oriented vertically and the other was via horizontal slices.

   The easier way was via horizontal slices, which we do first. The horizontal slices are approximated by disks of area πx^2 and thickness Δy. So the integral is

   $$\int_{-3}^{3} \pi x^2 dy$$

   Solving for x^2 gives

   $$x^2 = 4(1 - y^2/9),$$

   and we substitute this formula in the integral. That gives

   $$\int_{-3}^{3} 4\pi(1 - y^2/9)dy.$$

   Integrating this polynomial gives 16π.

   It was also possible to use cylinders. The formula for the volume of a cylindrical shell is $2\pi rh\Delta x$. Here $r = x$ and $h = 2\sqrt{9(1 - x^2/4)}$. Substituting in the integral, we get

   $$\int_0^2 2\pi x 2\sqrt{9(1 - x^2/4)}dx \tag{1}$$

   $$= 12\pi \int_0^2 x\sqrt{1 - x^2/4}\, dx \tag{2}$$

   $$\tag{3}$$

   At this point we do a u substitution, $u = 1 - x^2/4$, giving $xdx = -2du$. The bounds transform from 2 and 0 to 0 and 1. Thus the above integral is equal to

   $$12\pi \int_1^0 \sqrt{u} - 2du \tag{4}$$

   $$= 24\pi \int_0^1 \sqrt{u}du \tag{5}$$

   $$= 24\pi \frac{u^{3/2}}{3/2}\Big]_0^1 \tag{6}$$

   $$= 16\pi \tag{7}$$

   Many people revolved around the x-axis, which is incorrect.

   (b) The formula for the volume of a sphere is $\frac{4}{3}\pi r^3$. Setting this equal to the answer for 1(a), 16π, gives the cube root of twelve.

2. Problem 2

   Choices (b), (d), and (e) all give the area indicated.

   For (b) we slice the area into vertical rectangles, for (d) into horizontal rectangles, and for (e) we take the area under the arctangent graph and subtract the excess we have obtained.

3. Problem 3:

   (a) Set the $y-$axis through the cone so that $y = 0$ is at the bottom of the cone. Partition the interval $[0, 3]$ into n equal pieces each of length Δy. This slices the cone into circular disks, each of thicknessΔy.

   weight of the ith slice $\approx$ (volume of the disk) $\cdot$ (density on the slice) $\approx \pi r_1^2 \Delta y \cdot \rho(y_i)$

   Get r_i in terms of y_i by using similar triangles: $\frac{5}{3} = \frac{r_i}{y_i}$, so $r = \frac{5}{3}y_i$.

Therefore, weight of the ith slice $\approx \pi(\frac{5}{3}y_i)^2\Delta y \cdot \rho(y_i)$
Sum over all n slices to approximate the weight of the sludge.

$$\sum_{i=1}^{n}(25\pi/9)y_i^2\rho(y_i)\Delta y$$

(b)

$$\lim_{n\to\infty}\sum_{i=1}^{n}(25\pi/9)y_i^2\rho(y_i)\Delta y = (25\pi/9)\int_0^3 y^2\rho(y)dy$$

(c) Work = (Force) (distance) = (weight) (distance)
The ith slice has to move a distance $3 - y_i$ to get to the rim of the tank.
The work done in pumping out the ith slice $\approx \pi(\frac{5}{3}y_i)^2\Delta y \cdot \rho(y_i)(3 - y_i)$
Total work =

$$(25\pi/9)\int_0^3 y^2\rho(y)(3-y)dy$$

4. Problem 4

(a) $\int_0^\infty x\,dx : \lim_{b\to\infty}\int_0^b x\,dx = \lim_{b\to\infty}\frac{x^2}{2}|_0^b = \lim_{b\to\infty}\frac{b^2}{2}|_0^b = \infty$
Therefore, the integral diverges.

(b) item $\int_{-\infty}^{\infty}\sin x\,dx = \int_{-\infty}^{0}\sin x\,dx + \int_0^\infty \sin x\,dx$
Both of these integrals must converge in order for the original integral to converge.
$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty}\cos x|_0^b = \lim_{b\to\infty}\cos b - \cos 0 = \lim_{b\to\infty}\cos b - 1$ This limit does not exist ($\cos x$ oscillates ad infinitum.) Therefore $\int_0^\infty \sin x\,dx$ diverges (by oscillation) and hence our original improper integral diverges.

(c) Given only this information you cannot determine whether or not the improper integral converges. For instance, suppose $h(x) = \frac{1}{x}$. Then $\lim_{x\to\infty} h(x) = 0$ but
$\int_{100}^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_{100}^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln x|_{100}^b = \lim_{b\to\infty}\ln b - \ln 100 = \infty$. Therefore the improper integral diverges.
On the other hand, suppose $h(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} h(x) = 0$ but
$\int_{100}^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_{100}^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\frac{1}{x}|_{100}^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{100} = \frac{1}{100}$.

Addressing common errors: Any specific improper integral either converges or diverges. An integral that does not converge, by definition will diverge. Many students talked about the function $h(x)$ converging or diverging. That is not correct; it is the improper integral that converges or diverges. While it it true that $h(x)$ may be discontinuous somewhere on the interval $[100, \infty)$, and that may cause the improper integral to diverge, many people implied that this was the only condition under which the improper integral would diverge. That implication is not correct, as illustrated above.

5. Problem 5.

One could either chop up the distance or chop up the snake for this problem. For either one, though, we take x to represent the length of serpent hanging over the edge of the cliff.

First we show chopping the distance. The force from the hanging snake is $.3xg$, where g is the gravitational constant. The incremental amount of work done in pulling the snake from x to $x + \Delta x$ is approximated by $.3xg\Delta x$. The corresponding integral is

$$\int_0^5 .3xgdx = 3.75g$$

Now we chop the snake. A small section of the snake, Δx long, exerts a constant force of $.3g\Delta x$. If we say this section is x away from the edge of the cliff then the amount of work required to pull it up is $.3xg\Delta x$. Now we proceed as before.

Many people multiplied the density by some measure of length and then applied the first technique: this is incorrect because we need a density to multiply by the varying length of the hanging portion of the serpent.

As some sections did not discuss the necessity of multiplying by a gravitational constant, no points were taken off for its omission.

6. Problem 6

General comments: In each part, we must decide whether we want to use washers or shells, which also determines whether we integrate with respect to x or with respect to y. In part a), where we are revolving around an axis perpendicular to the x-axis, if we use shells we integrate with respect to x and if we use washers we integrate with respect to y. In part b) just the opposite is true: since we are revolving around an axis parallel to the x-axis, if we use washers we integrate with respect to x and if we use shells we integrate with respect to y.

(a) Say we decide to use shells (and integrate with respect to x). We need to know the quantities $H(x)$, $h(x)$ and $r(x)$ – the top height, the bottom height, and the radius respectively. By drawing a picture of the region, we find that $H(x) = \cos x$, $h(x) = 0$ (i.e. there is no bottom function here), and $r(x) = \pi - x$ (because it is the distance between x and π when x is less than or equal to π). The limits of integration extend from $-\pi/2$ to $\pi/2$, so the integral is

$$V = \int_{-\pi/2}^{\pi/2} (2\pi)(\pi - x)(\cos x)dx$$

Remark: It would *not* be correct to integrate from 0 to $\pi/2$ and double the answer: this is permissible to integrate even functions on an interval of the form $[-a, a]$, but although $\cos x$ is an even function, the entire integrand is not. Indeed, the fact that $(\pi - x)$ is larger on $[-\pi/2, 0]$ than on $[0, \pi/2]$ means that the integral over $[-\pi/2, 0]$ is larger than the integral over $[0, \pi/2]$ (or, more geometrically, this means that the radius is larger on $[-\pi/2, 0]$ than on $[0, \pi/2]$ so that the cylinders swept out will have larger volumes).
On the other hand, we could integrate with respect to y and use washers; now we need to find $R(y)$ and $r(y)$ the outer and inner radii respectively. We get $R(y) = \pi + \arccos y$ and $r(y) = \pi - \arccos y$, so

$$V = \int_0^1 \pi(\pi + \arccos y)^2 - (\pi - \arccos y)^2 dy$$

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(b) Say we use washers (hence integrate with respect to x). We get $R = 1$, $r = (1 - \cos x)$, so

$$V = \int_{-\pi/2}^{\pi/2} \pi(1^2 - (1 - \cos x)^2)dx.$$

(This time we *do* have symmetry about $x = 0$, so it would also be correct to integrate on $[0, \pi/2]$ and multiply by 2.)
Finally, we could integrate with respect to y and use shels: we need to find $H(y)$, $h(y)$ and $r(y)$. We have $H(y) = \arccos y$, $h(y) = -\arccos y$ and $r(y) = 1 - y$, so

$$V = \int_0^1 (2\pi)(\arccos y - (-\arccos y))(1 - y)dy.$$

7. Problem 7

The volume of the Tower is the integral from 0 to ∞ of the

cross-sectional area at height x:

$$\text{Volume} = \int_0^\infty xe^{-x}dx.$$

Integrating by parts with

$$\begin{array}{ll} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{array}$$

we see that
$\int xe^{-x}dx = \int udv = uv - \int vdu = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$
and therefore
Volume $= \int_0^\infty xe^{-x}dx = \lim_{t\to\infty} \int_0^t xe^{-x}dx = \lim_{t\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^t = \lim_{t\to\infty} \left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right)$

$$\lim_{t\to\infty} \frac{t}{e^t} == 0$$

You could do this with or without L'Hospital's rule. With it we have $\lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0$. Therefore

$$\boxed{\text{Volume} = 1}$$

8. (12 points) Consider the function $f(x) = \frac{C}{1+x^2}$ on the domain $(-\infty, \infty)$, where C is a constant. If C is chosen correctly then f is a probability density function.

(a) (4 points) Find the value of C such that $f(x)$ is a probability density function. For the rest of this problem, use this value of C.
Sol. A probability density fuction f must satisfy$\int_{-\infty}^\infty f(x)\,dx = 1$. (Since the domain of f(x) is $(-\infty, \infty)$, the random variable X *can* have negative value, so you can't say $\int_0^\infty f(x)\,dx = 1$.)
Therefore C should satisfy $C\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = 1$. While:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

And

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx \\ &= \lim_{a\to-\infty} \arctan x\Big|_a^0 \\ &= \arctan 0 - \lim_{a\to-\infty} \arctan a \\ &= 0 - (-\tfrac{\pi}{2}) \\ &= \tfrac{\pi}{2} \end{aligned}$$

You can use the same method or use the fact that $\frac{1}{1+x^2}$ is an even fuction to derive $\int_0^\infty \frac{1}{1+x^2}\,dx$ also equals $\frac{\pi}{2}$. So $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi$, $C\pi = 1$, $C = \frac{1}{\pi}$.

(b) (3 points)Find $P(-1 < X < 1)$.
Sol. Use $C = \frac{1}{\pi}$ (You can still get credit if you got the wrong C in (a) and use it here or (c),(d)):

$$\begin{aligned} P(-1 < x < 1) &= \int_{-1^1} \tfrac{1}{\pi(1+x^2)} \\ &= \tfrac{1}{\pi} \arctan x\Big|_{-1}^1 \\ &= \tfrac{1}{\pi}[\tfrac{\pi}{4} - (-\tfrac{\pi}{4})] \\ &= \tfrac{1}{2}. \end{aligned}$$

(The range of $\arctan x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$. So even though $\tan\frac{3\pi}{4}$ also equals -1, since $\frac{3\pi}{4}$ is outside the range of $\arctan x$, you should take $-\frac{\pi}{4}$ to be $\arctan(-1)$, not $\frac{3\pi}{4}$.)

(c) (2 points) The median of this distribution is the value a such that $\int_a^\infty f(x)\,dx = \frac{1}{2}$. Find the The median of this distribution is the value a such that $\int_a^\infty f(x)\,dx = \frac{1}{2}$. Find the median.

Sol. You can evaluate $\frac{1}{2} = \int_a^\infty \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2} - \frac{1}{\pi}\arctan a$ to get $\arctan a = 0$, $a = 0$.

You can also use the fact that $f(x)$ is even, so $\int_0^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx$ And since the total probability $\int_{-\infty}^\infty f(x)\,dx = 1$, $\int_0^\infty f(x)\,dx = 1/2$. Therefore $a = 0$. (In this argument, we didn't even use the value of C!)

(d) (3 points) $P(0 \leq X \leq \sqrt{3}) = \frac{1}{3}$. What is $P(\sqrt{3} \leq X)$?

Sol. To do this directly, calculate $\int_{\sqrt{3}}^\infty \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{6}$. Alternatively, you can use the fact that the median is 0, so $P(0 \leq X) = \frac{1}{2}$, $P(\sqrt{3} \leq X) = P(0 \leq X) - P(0 \leq X \leq \sqrt{3}) = 1/2 - 1/3 = 1/6$.