Problem Set 22

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n$ and

$-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2}x^n$. Hence, the equation $y' = x^2 y$ becomes

$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum_{n=2}^{\infty} c_{n-2}x^n = 0$ or $c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}]\, x^n = 0$. Equating coefficients

gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general

$c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{3^2 \cdot 2!}$,

$c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 6 \cdot 3} = \dfrac{c_0}{3^3 \cdot 3!}, \ldots$, and $c_{3n} = \dfrac{c_0}{3^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}$$

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $3xy'(x) = 3x \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} 3nc_n x^n$,

$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$, and the equation

$y'' + 3xy' + 3y = 0$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} 3nc_n x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0 \quad \Leftrightarrow$

$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + 3nc_n + 3c_n]x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{-3nc_n - 3c_n}{(n+2)(n+1)} = \dfrac{-3c_n(n+1)}{(n+2)(n+1)} = -\dfrac{3c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_2 = -\dfrac{3c_0}{2}$,

$c_4 = -\dfrac{3c_2}{4} = (-1)^2 \dfrac{3^2 c_0}{2^2 \cdot 2!}$, $c_6 = -\dfrac{3c_4}{6} = (-1)^3 \dfrac{3^3 c_0}{2^3 \cdot 3!}, \ldots$, $c_{2n} = (-1)^n \dfrac{3^n c_0}{2^n n!}$ or, equivalently, $c_0 \left(-\dfrac{3}{2}\right)^n \dfrac{1}{n!}$.

Also, $c_3 = -\dfrac{3c_1}{3}$, $c_5 = -\dfrac{3c_3}{5} = (-1)^2 \dfrac{3^2 c_1}{5 \cdot 3}$, $c_7 = -\dfrac{3c_5}{7} = (-1)^3 \dfrac{3^3 c_1}{7 \cdot 5 \cdot 3}, \ldots$,

$c_{2n+1} = (-1)^n \dfrac{3^n c_1}{(2n+1)(2n-1) \cdot \cdots \cdot 5 \cdot 3}$. Since $(2n+1)(2n-1) \cdot \cdots \cdot 5 \cdot 3$ can be written as

$$\frac{(2n+1)\ (2n)\ (2n-1)\ (2n-2) \cdot \cdots \cdot 5 \cdot 4 \cdot 3 \cdot 2}{(2 \cdot n) \cdot [2(n-1)] \cdot (2 \cdot 2) \cdot (2 \cdot 1)} = \frac{(2n+1)!}{2^n \cdot n!},$$

c_{2n+1} can be written as $(-1)^n \dfrac{3^n c_1 2^n n!}{(2n+1)!} = c_1 \dfrac{(-6)^n n!}{(2n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n \frac{1}{n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-6)^n n!}{(2n+1)!} x^{2n+1}$$

Note that the c_0-term can be written as $c_0 \sum_{n=0}^{\infty} \left(-\dfrac{3x^2}{2}\right)^n \dfrac{1}{n!} = c_0 e^{-3x^2/2}$.