Section 6.7 #2, 3, 4, 6, 8; Chapter 6 Review #29; Extra Credit 6.7 #13

Section 6.7

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)dt$.
(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)dt$.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example I, namely, (1) $f(x) \geq 0$for all x, and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$.Since $f(x) = 0$or $f(x) = 0.1$, condition (1) is satisfied. For condtion (2), we see that $\int_{-\infty}^{\infty} f(x)dx = \int_0^{10} 0.1dx = [\frac{(1}{10)}x]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.
(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the meant to be halfway between the endpoints of the interval; that is, $x = 5$.

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{10} x(0.1)dx = [\frac{1}{20}x^2]_0^{10} = \frac{100}{20} = 5,$$

as expected.

4. (a) As in the preceding exercise, (1) $f(x) \geq 0$and (2) $\int_{-\infty}^{\infty} f(x)dx = \int_0^{10} f(x)dx = (0.5)(10)(0.2) = 1[area\, of\, a\, triangle]$. So $f(x)$ is a probability density function.
(b) (i) $P(X < 3) = \int_0^3 f(x)dx = (0.5)(3)(0.1) = \frac{3}{20} = 0.15$
(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).

$$P(X > 8) = \int_8^{10} f(x)dx = \frac{1}{2}(2)(0.1) = \frac{2}{20} = 0.1,$$

so $P(3 \leq X \leq 8) = 1 - 0.15 - 0.1 = 0.75$.
(c) We find equations of the lines from (0,0) to (6, 0.2) and from (6, 0.2) to (10, 0), and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if} 0 \leq x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if} 6 \leq x < 10 \\ 0 & otherwise \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^6 x(\frac{1}{30}x)dx + \int_6^{10} x(-\frac{1}{20}x + \frac{1}{2})dx = [\frac{1}{90}x^3]_0^6 + [-\frac{1}{60}x^3 + \frac{1}{4}x^2]_6^{10}$$

$$= \frac{216}{90} + (-\frac{1000}{60} + \frac{100}{4}) - (-\frac{216}{60} + \frac{36}{4}) = \frac{16}{3} = 5.\overline{3}$$

6. (a)

$$\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

(i)

$$P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}dt = [-e^{-t/1000}]_0^{200} = -e^{-1/5} + 1 \approx 0.181$$

(ii)

$$P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t/1000}dt = \lim_{x\to\infty}[-e^{-t/1000}]_{800}^{x} = 0 + e^{-4/5} \approx 0.449$$

(b) We need to find m so that $\int_m^\infty f(t)dt = 0.5 \Rightarrow \int_m^\infty \frac{1}{1000}e^{-t/1000}dt = 0.5 \Rightarrow$ $\lim_{x\to\infty}[-e^{-t/1000}]_m^x = 0.5$
$\Rightarrow 0 + e^{-m/1000} = 0.5 \Rightarrow -m/1000 = \ln 0.5 \Rightarrow m = -1000(\ln 0.5) = 1000 \ln 2 \approx$ 693.1h.

8. (a) With $\mu = 69$ and $\sigma = 2.8$, we have

$$P(65 \leq X \leq 73) = \int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} exp\left(-\frac{(x-69)^2}{2(2.8^2)}\right) dx \approx 0.847$$

(using a calculator or computer to estimate the integral).
(b)

$$P(X > 6feet) = P(X > 72in) = 1 - P(0 \leq X \leq 72) \approx 1 - 0.858 = 0.142,$$

sp 14.2% of the adult male population is more than 6 feet tall.

Chapter 6 Review

29. (a) The probability density function is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$$

$$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{8}e^{-t/8}dt = [-e^{-t/8}]_0^3 = -e^{3/8} + 1 \approx 0.3127$$

(b)

$$P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}dt = \lim_{x\to\infty}[-e^{-t/8}]_{10}^{x} = 0 + e^{-5/4} \approx 0.2865$$

(c) We need to find m so that $P(X \geq m) = 0.5 \Rightarrow \int_m^\infty \frac{1}{8}e^{-t/8}dt = 0.5 \Rightarrow$ $\lim_{x\to\infty}[-e^{-t/8}]_m^x = 0.5$
$\Rightarrow 0 + e^{-m/8} = 0.5 \Rightarrow -m/8 = \ln 0.5 \Rightarrow m = -8(\ln 0.5) = 8 \ln 2 \approx 5.55$minutes.

Extra Credit (section 6.7)

13. (a) First

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0 \text{for} r \geq 0$$

Next,

$$\int_{-\infty}^{\infty} p(r)dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0} dr$$

Here, the solution manual says to use a result from an unassigned problem (#12), which in turn says to use a computer or a table of intergrals, as well as L'Hôpital's Rule, to get that the intergral equals 1, therby satisfying the second condition for a function to be a probability density function.

(b) Using L'Hôpital's Rule, $\frac{4}{a_0^3} \lim_{r\to\infty} r^2 e^{-2r/a_0} = \frac{4}{a_0^3} \lim_{r\to\infty} 2r(a_0/2)e^{-2r/a_0} = 2/a_0^2 \lim_{r\to\infty} 2(a_0/2)e^{-2r/a_0} = 0$.

To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3} \left[\ r^2 e^{-2r/a_0}(-\tfrac{2}{a_0}) + e^{-2r/a_0}(2r) \ \right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)(-\frac{r}{a_0} + 1)$$

$$p'(r) = 0 \Leftrightarrow r = 0 \text{ or } 1 = \frac{r}{a_0} \Leftrightarrow r = a_0 [a_0 \approx 5.59\text{x}10^{-11}\text{m}]$$

$p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum helps.

$$p(a_0) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9,684,098,979$$

With a maximum of nearly 10 billion and a total area under the curve of 1, we know the "hump" in the graph must be extremely narrow.

(d)

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds \Rightarrow P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds$$

Using (2) from the solution to exercise 12 (or perhaps a computer), $P(4a_0) \approx 0.986$

(e) $\mu = \int_{-\infty}^{\infty} rp(r)dr = 4/a_0^3 \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0} dr$. Integrating by parts three times, we get that the integral equals $4/\text{a}_0^3[(a_0^4/16)(-6)] = (3/2)a_0$.