Problem Set # 8 (Section 8.7)

12. $f(x) = \sqrt{x}$, $a = 4$

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	3×2^{-8}
4	$-\frac{15}{16}x^{-7/2}$	-15×2^{-11}
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(4) = \frac{(-1)^{n-1} \times 1 \times 3 \times 5 \times \ldots \times (2n-3)}{2^{3n-1}} \quad , \text{ for } n \geq 2$$

$$\Rightarrow \sqrt{x} = 2 + \frac{x-4}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \times 1 \times 3 \times 5 \times \ldots \times (2n-3)}{n! \times 2^{3n-1}} (x-4)^n$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1 \times 3 \times 5 \times \ldots (2n-3)(2n-1)(x-4)^n (x-4)}{n!(n+1)\, 2^{3n+2}} \times \frac{2^{3n-1} \times n!}{1 \times 3 \times 5 \times \ldots \times (2n-3)(x-4)^n} \right|$$

$$= \frac{|x-4|}{8} \lim_{n\to\infty} \left| \frac{(2n-1)}{n+1} \right| = \frac{|x-4|}{4} < 1 \text{ for convergence}$$

$$\Rightarrow |x-4| < 4 \Rightarrow \underline{\underline{R = 4}}$$

14. $f(x) = \cos(x)$, $a = -\frac{\pi}{4}$

n	$f^{(n)}(x)$	$f^{(n)}\left(-\frac{\pi}{4}\right)$
0	$\cos(x)$	$\sqrt{2}/2$
1	$-\sin(x)$	$\sqrt{2}/2$
2	$-\cos(x)$	$-\sqrt{2}/2$
3	$\sin(x)$	$-\sqrt{2}/2$
4	$\cos(x)$	$\sqrt{2}/2$

$$\Rightarrow \cos(x) = f\left(-\frac{\pi}{4}\right) + f'\left(-\frac{\pi}{4}\right)\left(x+\frac{\pi}{4}\right) + \frac{f''\left(-\frac{\pi}{4}\right)}{2!}\left(x+\frac{\pi}{4}\right)^2 + \frac{f^{(3)}\left(-\frac{\pi}{4}\right)}{3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(-\frac{\pi}{4}\right)}{4!}\left(x+\frac{\pi}{4}\right)^4 + \dots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) - \frac{1}{2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x+\frac{\pi}{4}\right)^4 + \dots\right]$$

$$= \frac{\sqrt{2}}{2}\left[1 - \frac{1}{2!}\left(x+\frac{\pi}{4}\right)^2 + \frac{1}{4!}\left(x+\frac{\pi}{4}\right)^4 - \dots\right] + \frac{\sqrt{2}}{2}\left[\left(x+\frac{\pi}{4}\right) - \frac{1}{3!}\left(x+\frac{\pi}{4}\right)^3 + \dots\right]$$

$$= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2n!}\left(x+\frac{\pi}{4}\right)^{2n} + \frac{1}{(2n+1)!}\left(x+\frac{\pi}{4}\right)^{2n+1}\right]$$

Rewrite as $\frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2} \times \left(x+\frac{\pi}{4}\right)^n}{n!}$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(x+\frac{\pi}{4}\right)^n\left(x+\frac{\pi}{4}\right)}{(n+1)\,n!} \times \frac{n!}{\left(x+\frac{\pi}{4}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{\left(x+\frac{\pi}{4}\right)}{n+1}\right| = 0 < 1$$

$\Rightarrow R = \infty$

18. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} \Rightarrow f(x) = \sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n \times n!}x^n$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x)^n \times x}{2 \cdot 2^n \times (n+1) \times n!} \times \frac{2^n \times n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2 \times (n+1)}\right| = 0 < 1$$

$\Rightarrow \underline{R = \infty}$

20. $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \dfrac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{8n+4}}{(2n+1)!}$

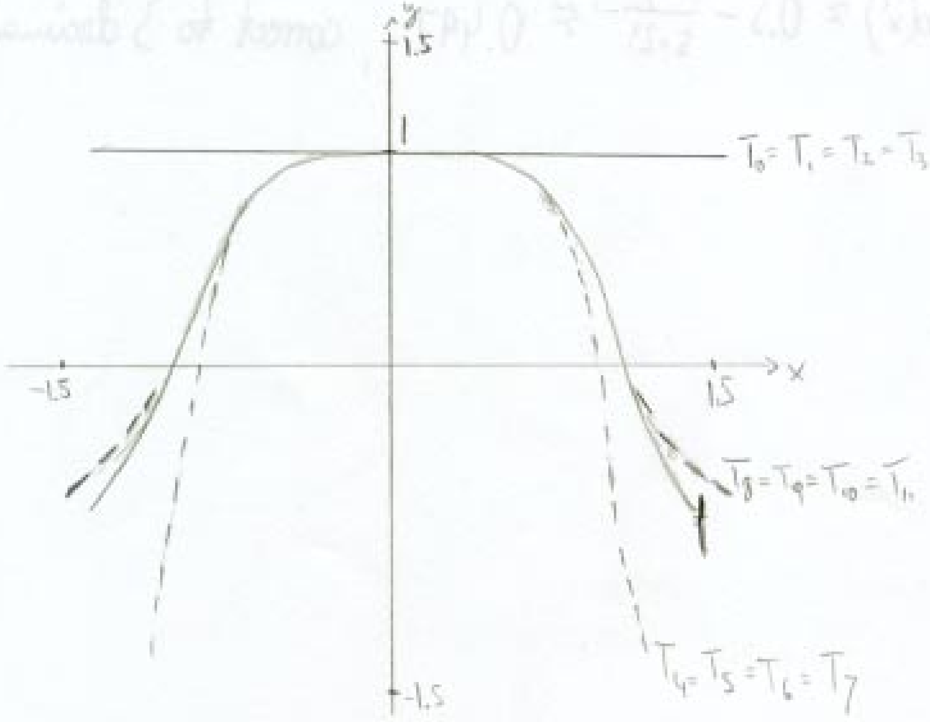
$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{8n+12}}{(2n+1)!(2n+2)(2n+3)} \times \frac{(2n+1)!}{x^{8n+4}} \right| = \lim_{n\to\infty} \left| \frac{x^8}{(2n+2)(2n+3)} \right| = 0 < 1$$

$\Rightarrow \underline{R = \infty}$

23. $\sin^2(x) = \frac{1}{2}\left[1 - \cos(2x)\right] = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{2n!}\right] = 2^{-1}\left[1 - 1 - \sum_{n=1}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{2n!}\right]$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \times 2^{2n-1} \times x^{2n}}{2n!}, \quad R = \infty \text{ (by Ratio Test)}$$

27. $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^2)^{2n}}{2n!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n}}{2n!}, \ R = \infty$



As n increases, $T_n(x)$ becomes a better approximation to $f(x)$

29. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\Rightarrow$ Thus $e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2) + \frac{1}{6!}(0.2)^6 - \ldots$

$\Rightarrow$ Since $\frac{1}{6!}(0.2)^6 = 8.\bar{8} \times 10^{-8}$, by the Alternating Series Theorem,

$e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!}$, correct to 5 decimal places.

36. $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{2n!}$

$$\int_0^{0.5} \cos(x^2)\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2n!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+1}}{(4n+1)\times(2n)!} \right]_0^{0.5} = 0.5 - \frac{(0.5)^5}{5 \times 2!} + \frac{(0.5)^9}{9 \times 4!} - \ldots$$

$\Rightarrow$ Since $\frac{(0.5)^9}{9 \times 4!} \approx 0.000009$, by the Alternating Series Estimation Theorem $\int_0^{0.5} \cos(x^2) \approx 0.5 - \frac{(0.5)^5}{5 \times 2!} \approx 0.497$, correct to 3 decimal places.