Problem Set #33

20. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

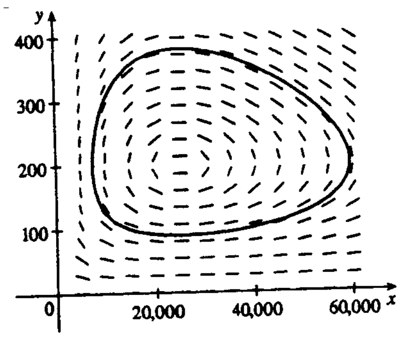
(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\begin{cases} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{cases} \Rightarrow \begin{cases} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{cases} \Rightarrow y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

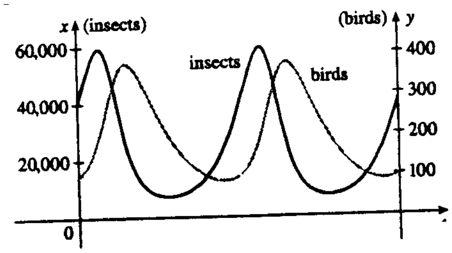
(c) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$$

(d)



At $(x, y) = (40{,}000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59{,}646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e)



Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

21. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0$ $\Leftrightarrow$ $x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\begin{cases} 0 = 0.4x(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{cases} \Rightarrow \begin{cases} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{cases}$$

The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4(25{,}000)[(1 - 0.000005 \cdot 25{,}000) - 0.005y]$ $\Rightarrow$ $0 = 10{,}000[(1 - 0.125) - 0.005y]$ $\Rightarrow$ $0 = 8750 - 50y$ $\Rightarrow$ $y = 175$.

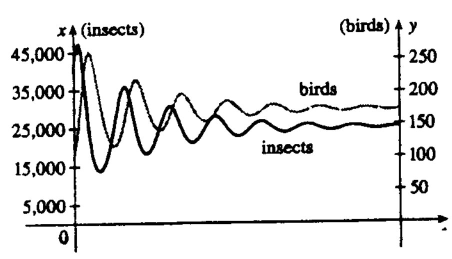
Case (i): $y = 0$, $x = 0$: Zero populations

Case (ii): $y = 0$, $x = 200{,}000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: The predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d)



Handout H

1.) If y_1 y_2 are solutions

$$\Rightarrow y_1 + by_1 + cy_1 \quad 0$$

$$y_2 + by_2 + cy_2 = 0$$

Consider $y \quad C_1y_1 + C_2y_2$

$$\Rightarrow C_1y_1'' + C_2y_2 + bC_1y + bC_2y_2' + cC_1y_1 + cC_2y_2$$

$$= C_1(\underbrace{y_1'' + by_1' + cy_1}) + C_2(\underbrace{y_2'' + by_2' + cy_2}_{= 0})$$

$$C_1(0) + C_2(0)$$

$$= 0$$

$y \quad Cy_1 + C_2y_2$ is a solution

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28) $y = e^{\lambda t}$; $y'' + 7y' + 12y = 0$

a) $\Rightarrow \lambda^2 e^{\lambda t} + 7\lambda e^{\lambda t} + 12e^{\lambda t} = 0$ (plugging in)

$\Rightarrow \lambda^2 + 7\lambda + 12 = 0$

$\Rightarrow (\lambda + 4)(\lambda + 3) = 0$

$\therefore \lambda_1 = -4$

$\lambda_2 = -3$

b) $y = C_1 e^{-4t} + C_2 e^{-3t}$

$\Rightarrow (16C_1 e^{-4t} + 9C_2 e^{-3t}) + (-28C_1 e^{-4t} - 21C_2 e^{-3t}) + (12C_1 e^{-4t} + 12C_2 e^{-3t})$

$= \underbrace{(16C_1 e^{-4t} - 28C_1 e^{-4t} + 12C_1 e^{-4t})}_{=0} + \underbrace{(9C_2 e^{-3t} - 21C_2 e^{-3t} + 12C_2 e^{-3t})}_{=0}$

$= 0 + 0$

$= 0$

$\Rightarrow$ $y = C_1 e^{-4t} + C_2 e^{-3t}$ is a solution

29) $y = e^{\lambda t}$; $y'' + 4y' + 4y = 0$

a) $\Rightarrow \lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4e^{\lambda t} = 0$ (plugging in)

$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow (\lambda + 2)(\lambda + 2) = 0$

$\therefore \lambda = -2$

b) $y = te^{\lambda t} = te^{-2t}$

$\Rightarrow (te^{-2t})'' + 4(te^{-2t})' + 4te^{-2t}$ (use product rule)

$= (-2te^{-2t} + e^{-2t})' + 4(e^{-2t} - 2te^{-2t}) + 4te^{-2t}$

$= (4te^{-2t} - 2e^{-2t} + e^{-2t}(-2)) + 4e^{-2t} - 8te^{-2t} + 4te^{-2t}$

$= (4te^{-2t} - 8te^{-2t} + 4te^{-2t}) + (-2e^{-2t} - 2e^{-2t} + 4e^{-2t})$

$= 0 + 0$

$= \underline{0}$

$\therefore$ $\underline{y = te^{-2t} \text{ is a solution}}$