Mathematics 1b - Solution Set for PS 3
Problem Set # 3

Do: §8.3 # 1, 2, 3, 4, 10, 19, 20.

1) The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}} dx$, $a3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}} dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}} dx$. The integral converges because it is a p-series with $p = 1.3 > 1$, so the series converges.

2) From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$. Therefore, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$.

3a) We cannot say anything because $\sum a_n$ is not bound above by a series known to converge. Therefore, $\sum a_n$ can either converge or diverge. (Answers can vary - use your judgment.)
3b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent because it is always positive and bound by $\sum b_n$, a series that has greater partial sums and that converges. (part (i) of Comparison Test)

4a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. (part (ii) of Comparison Text)
4b) We cannot say anything about $\sum a_n$ because it is not bound below by a series known to diverge. Therefore, $\sum a_n$ can either converge or diverge. (Again, use your judgment.)

10) $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2}\frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, a series that diverges and has lower corresponding terms. (part (ii) of Comp. Test)

19) $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. (part (ii) of Comp. Test)

20) $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = (\frac{3}{2})^n$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with $\sum_{n=1}^{\infty} (\frac{3}{2})^n$, a geometric series that diverges ($r = \frac{3}{2} > 1$).

Plus:

Problem on p−series

In this problem you will learn about a family of series known as $p-$series. A $p-$series is a series of the form $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$.

1. If $p < 0$ then the series diverges by the nth term test. Explain.

   I am actually not sure what the nth term test is, but if $p < 0$, the series would be the same as $\sum_{n=1}^{\infty} n^{-p}$ (with -p being positive), which diverges based on the Test for Divergence because $\lim_{n\to\infty} n^{positivevalue} \neq 0$.

2. Show that if $p > 1$ then $\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx$ is finite.

   $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \lim_{b\to\infty} \frac{-1}{(p-1)x^{p-1}}|_1^b = \lim_{b\to\infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)}$
   If $(p-1) < 0 or p < 1$, then the above integral would approach infinity because the exponent of b would be negative and would bring b to the numerator (with a positive exponent). This would approach infinity as b approaches infinity.

   If $(p-1) = 0 or p = 1$, then the integral of $\frac{1}{x}$ would be $\lim_{b\to\infty} \ln x|_1^b$, which is infinite (because $\lim_{b\to\infty} \ln b = \infty$).

If $(p-1) > 0 or p > 1$, the definite integral is finite because b remains in the denominator and causing that term to approach zero as b approaches infinity. Thus, if $p > 1$, the definite integral is finite.

(NOTE: Students only need to show the last third, since that is all that was asked. I gave more in case some students explain all ranges.)

Show that if $0 < p < 1$ then $\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \infty$.

Along the same lines as above, if $0 < p < 1$, then $(p-1) < 0$ and $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \infty$. Thus, $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \lim_{b\to\infty} \frac{-1}{(p-1)x^{p-1}}|_1^b = \lim_{b\to\infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)} = \infty - \frac{-1}{(p-1)(1)}$. Because the second term is finite, the difference is infinite. Thus, $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \infty$ if $0 < p < 1$.

(You'll need to break this into two cases. Think about why.)

I don't think that we need to split it into two cases - am I missing something?)

Conclude that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for $0 < p < 1$ and converges for $p > 1$.

Based on the Integral Test, $\int_1^\infty \frac{1}{x^p}$ converges when $p > 1$ because $\int_1^\infty \frac{1}{x^p}$ is convergent for those values of p. Likewise, $\int_1^\infty \frac{1}{x^p}$ diverges when $0 < p < 1$ because $\int_1^\infty \frac{1}{x^p}$ is divergent for those values of p.

3. Conclude from your work in parts (a) and (b) that $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p > 1$ and diverges if $p < 1$.

It was shown above that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p < 0$ and if $0 < p < 1$. If $p = 0$, the series becomes $\sum_{n=1}^{\infty} 1$, which diverges as well due to the Divergence Test. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p < 1$ and converges if $p > 1$.