Mathematics 1b - Solution Set for PS 10
Problem Set # 10

Do: §8.9 # 3, 19, 20, 22, page 641 # 40

3) $f(x) = \ln x, a = 1, n = 4$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$\frac{1}{x}$	1
2	$-\frac{1}{x^2}$	-1
3	$\frac{2}{x^3}$	2
4	$-\frac{6}{x^4}$	-6

$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$
For graph, see solutions manual p. 631.

19) All derivatives of e^x are e^x, so
$|R_4(x)| \le \frac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$,
$R_n(0.1) \le \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$,
and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus we need four terms of the MacLaurin series ($n = 0, 1, 2, 3$). (in fact the sum is $1.10516\bar{6}$ and $e^{0.1} \approx 1.10517$.)

20) The Maclaurin series for $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x)^n}{n}$ for $|x| < 1$.
$\ln(1.4) = \ln(1+0.4) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.4)^n}{n}$. This is an alternating series. Using the Alternating Series Estimation Theorem,
$|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (nonzero) terms of the Maclaurin series.

22) $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$.
Using the Alternating Series Estimation Theorem:
$|-\frac{1}{6!}x^6| < 0.005 \Rightarrow |x| < (3.6)^{1/6} \approx 1.238$. Graph: One should graph $y = \cos(x) + 0.005$ and $y = \cos(x) - 0.005$, and see where $y = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$ crosses outside (see picture p. 636 of solutions manual). Since cosine and our approximation function are even, need only to check $x > 0$. Range: $-1.238 < x < 1.238$.

Chapter 8 Exercises p 641: 40) Use the Maclaurin series for e^x, which we know:
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^{2x} = x\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$.
The radius of convergence for the Maclaurin series for e^x is $R = \infty$; it is the same for this series.

Plus:

Handout C

1) $f(x) = x^{1/3}$ at $x = 27$.
(a)

n	$f^{(n)}(x)$	$f^{(n)}(27)$
0	$x^{1/3}$	3
1	$\frac{1}{3}x^{-2/3}$	$1/3^3 = 1/27$
2	$\frac{1}{3}(-\frac{2}{3})x^{-5/3}$	$-2/3^7$
3	$\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})x^{-8/3}$	$10/3^{11}$
4	$\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})x^{-11/3}$	$-80/3^{15}$

$T_3(x) = 3 + \frac{1}{27}(x-27) + \frac{(-2/3^7)}{2!}(x-27)^2 + \frac{(10/3^{11})}{3!}(x-27)^3$
$T_3(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{3^7}(x-27)^2 + \frac{5}{3^{12}}(x-27)^3$
(b)
$28^{1/3} \approx T_3(28) = 3 + \frac{1}{27}(1) - \frac{1}{3^7}(1)^2 + \frac{5}{3^{12}}(1)^3$
$= 3 + \frac{1}{27} - \frac{1}{3^7} + \frac{5}{3^{12}} \approx 3.03658920$.
(c) $|\text{Error}| \le |a_4| = |\frac{-80/3^{15}}{4!}(1)^4| \approx 2.32306 \times 10^{-7}$
(d)

For $27 \le x \le 28$, $|f^{(4)}(x)| \le |f^{(4)}(27)| = 80/3^{15}$. So $M = 80/3^{15}$ and by Taylor's Inequality, $|R_4(28)| \le \frac{M}{4!}(1)^4 \approx 2.32306 \times 10^{-7}$.

2) $\ln(1+u), -1 < u \le 1$. (a) As in problem 20 above, The Maclaurin series for $\ln(1+u)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(u)^n}{n}$. Among other methods, this can be derived from $\frac{1}{1-(-u)}$ by integration.
(b) Letting $u = x - 1$, a power series for $\ln(x)$ centered at $x = 1$ is
$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.
(c)
As in problem 3 above,

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$\frac{1}{x}$	1
2	$\frac{(-1)}{x^2}$	-1
3	$\frac{(-1)(-2)}{x^3}$	2
4	$\frac{(-1)(-2)(-3)}{x^4}$	-6

We can see that for $n \ge 1$, $f^{(n)}(1) = \frac{(-1)^{n+1}(n-1)!}{(1)^n}$. $f(1) = 0$, so the 0th term is 0. So the Taylor Series for $\ln(x)$ at $x = 1$ is
$ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.
This agrees with the answer in part (b).