Exam 1 Review Part II

Math 1B October '99

continuing on...

- Power Series!
  - (1) - Examples / Definition
  - (2) - Convergence (Radius of..., Interval of...)
  - (3) - Differentiation / Anti-differentiation
- Taylor / Maclaurin Series!
  - (4) - Definition / Formula
  - (5) - Examples to know
  - (6) - Finding by tricks/substitutions
  - (7) - Accuracy of certain approximations

(1) Power Series
general form:

for values of x when a power series converges the power series is an actual function of x

Basic question to be answered - when does a power series converge (i.e. for what x)

Example 1 which of the following are power series?

(a) $1 + 3x + 5x^5 + 9x^9 + 100x^{100} + 101x^{101} + 102x^{102} + \cdots$

(b) $(x-\pi) - \frac{1}{5}(x-\pi)^5 + \frac{1}{9}(x-\pi)^9 - \cdots$

(c) $x + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$

(d) $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

(e) $\frac{\pi}{x} + \pi + \pi x + \frac{\pi}{2}x^2 + \frac{\pi}{3!}x^3 + \cdots$

(2) Convergence!
To find which values of x lead to convergence for a power series typically your best shot is...

(2) continued

The Ratio Test. Given $\sum_{k=1}^{\infty} a_k(x-c)^k$ calculate

$\lim_{k\to\infty} \left| \frac{a_{k+1}(x-c)^{k+1}}{a_k(x-c)^k} \right|$ this needs to be < 1 for convergence ($> 1 \Rightarrow$ diverges)

remember! you need to check the endpoints of the interval you come up with separately

so you might get to $|x-c| < M$

and so... $c-M < x < c+M$

check endpoints $x = c-M$, $x = c+M$

Resulting values of x when power series converges is called the

Half of the length of ↑ is called the

Example 2 Find R.O.C/IOC for $\sum_{k=1}^{\infty} \frac{k \cdot 2^k}{2k+3} x^k$

Example 3 Find ROC/IOC for $\sum_{k=1}^{\infty} \frac{k^2}{k!} (x-10)^{2k}$

Example 4 Find ROC/IOC for $x + 3x^2 + 5x^3 + \cdots + (2k-1) \cdot x^k + \cdots$

Example 5 Find ROC/IOC for $1 + x + x^2 + \cdots + x^k + \cdots$

(3) Differentiation / Anti-differentiation

if $f(x) = \sum_{k=1}^{\infty} a_k (x-c)^k$ then can find $f'(x)$ by taking derivative term-by-term (at least for where it converges)

$f'(x) =$

Likewise can integrate term by term to find a function $g(x)$ with $g'(x) = f(x)$ (an anti-derivative)

if $f(x) = \sum_{k=1}^{\infty} a_k (x-c)^k$ then $g(x) =$

Useful Tip → the Radius of Convergence is the <u>same</u> for all of these power series $f(x)$, $f'(x)$, and $g(x)$ although <u>!Note!</u> the <u>endpoints</u> might change (in terms of convergence)

<u>Example</u> 6 if $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ find $f'(x)$ (the first four non-zero terms)

<u>Example</u> 7 find an anti-derivative $g(x)$ for $f(x)$ in Ex. 6, with $g(0) = $:

<u>Example</u> 8 Find derivative / anti-derivative of $f(x) = \sum_{k=1}^{\infty} \frac{(k+1)}{3k+1} x^k$ (let value of antiderivative be 0 for $x=0$)

(4) Taylor / Maclaurin Series

Know formula Taylor Series for $f(x)$ around $x=c$ is:

around $x=0$: (also known as a Maclaurin Series)

(Take first n terms – get n^{th} degree Taylor or Maclaurin Polynomial)

This is the best approximation to $f(x)$ by an n^{th} degree polynomial

Example 9 find best cubic approximation around $x=0$ for $f(x) = \cos(x) + 2x + 1$

Example 10 find first 4 non-zero terms of the Maclaurin Series for $f(x) = \sqrt[4]{x+2}$

Example 11 find first 4 terms of Taylor Series for $\sqrt[4]{x+2}$ around $x = 14$

(4) continued

Note – you'd like to find $f(c)$, $f'(c)$, $f''(c)$, etc. as easily as possible, so if possible pick an "easy" c

Example 12 You'd to find a Taylor series for $\cos(x)$ for c $\cos(x)$ for values of x near to value of c would be a good one to this?

(5) Know your basic examples by heart!

$\sin x =$

$\cos x =$

$e^x =$

$\dfrac{}{1-x} =$

(6) Avoid work at all costs! Strive for shortcuts!

Example 13 Find the first four non-zero terms the Maclaurin series for $\sin(x^2)+\cos(x^2)$

Example 14 Find the first four non-zero terms of the Maclaurin series for $f(x) = \dfrac{x^4}{3} - 3x^2 + 73$

(7) Accuracy of approximations using alternating series

Sometimes, when you're using a power series which leads to an alternating series which passes the alternating series test, you can take advantage of the alternating series trick:

<u>Ex.</u> If you estimate the sum of $1-\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^3+\cdots$
$$= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k$$

by summing up the first four terms $1-\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^3 = \frac{5}{8}$

Then your approximation is within $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ of the actual sum (which, of course, is just $\frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$)

<u>Example 15</u> Use as many terms as necessary from the Maclaurin series for $\sin(x)$ to approximate $\sin(1)$ to 1 decimal place of accuracy (i.e. so that the error has absolute value $< .05 = \frac{1}{20}$)

Answers to Example Questions

Example 1 (a) yes (b) yes (c) no (d) no - has negative powers of x (e) no - same reason

Example 2 $\lim_{k\to\infty}\left|\frac{(k+1)2^{k+1}x^{k+1}}{(2(k+1)+3)}\Big/\frac{k\cdot 2^k}{2k+3}x^k\right| = \lim_{k\to\infty}\left|\left(\frac{k+1}{k}\right)\left(\frac{2k+3}{2k+5}\right)\cdot 2x\right| = 2|x| < 1$

means $-\frac{1}{2} < x < \frac{1}{2}$

check $x = +\frac{1}{2}$: $\sum_{k=1}^{\infty}\frac{k\cdot 2^k}{2k+3}\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty}\frac{k}{2k+3}$ diverges (divergence test)

$x = -\frac{1}{2}$: get $\sum_{k=1}^{\infty}\frac{k\cdot 2^k}{2k+3}\left(\frac{-1}{2}\right)^k = \sum_{k=1}^{\infty}\frac{(-1)^k k}{2k+3}$ diverges as well

so Interval of Convergence is $-\frac{1}{2} < x < \frac{1}{2}$, Radius of C. $= \frac{1}{2}$

Example 3 $\lim_{k\to\infty}\left|\frac{(k+1)^2}{(k+1)!}(x-10)^{2(k+1)}\Big/\frac{k^2}{k!}(x-10)^{2k}\right| = \lim_{k\to\infty}\left|\left(\frac{k+1}{k}\right)^2\frac{1}{k+1}(x-10)^2\right| = 0$

because of this term

so converges for all x

(you can say then that the radius of conv. is ∞)

Example 4 $x + 3x^2 + 5x^3 + \cdots + (2k-1)x^k + \cdots$ get $\lim_{k\to\infty}\left|\frac{(2(k-1)+1)x^{k+1}}{(2k-1)x^k}\right| = |x| < 1$

so $-1 < x < 1$,

check endpoints, $x=1$ get $1+3+5+\cdots$ diverges!

$x=-1$ get $-1+3-5+\cdots$ diverges as well

so interval of convergence is $-1 < x < 1$, radius $= 1$

Example 5 $1 + x + x^2 + \cdots$ it's just a geom. series! with ratio x,

converges when $|x| < 1$

so interval is $-1 < x < 1$, radius $= 1$

Example 6 $f'(x) = 1 - 3\frac{x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

makes sense, since $f(x) = \sin x$, $f'(x) = \cos x$

Example 7 $g(x) = c + \frac{x^2}{2} - \frac{x^4}{4\cdot 3!} + \frac{x^6}{6\cdot 5!} - \frac{x^8}{8\cdot 7!} + \cdots$ now when $x=0$,

$g(0) = c = 3$ so $c = 3$

$g(x) = 3 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$ (same as $4 - \cos x$)

Example 8 $f'(x) = \sum_{k=1}^{\infty}\frac{k(k+1)x^{k-1}}{3k+1}$ antiderivative $c + \sum_{k=1}^{\infty}\frac{(k+1)\,x^{k+1}}{(3k+1)(k+1)} = \sum_{k=1}^{\infty}\frac{x^{k+1}}{(3k+1)}$

(when $x=0 \Rightarrow c=0$)

Example 9 $f(x) = \cos(x) + 2x + 1$ at $x=0$: $f(0) = 1 + 0 + 1 = 2$

$f'(x) = -\sin(x) + 2$ $\quad f'(0) = 2$

$f''(x) = -\cos(x)$ $\quad f''(0) = -1$

$f'''(x) = \sin(x)$ $\quad f'''(0) = 0$

$\vdots$

so using $f(c) + f'(c)\cdot(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$

best cubic polynomial approximation is

$$2 + 2x + \frac{-1}{2}x^2 + \frac{0}{3!}x^3 = 2 + 2x - \frac{x^2}{2}$$

Example 10 $f(x) = (x+2)^{1/4}$ at $x=0$ $\quad f(0) = 2^{1/4}$

$f'(x) = \frac{1}{4}(x+2)^{-3/4}$ $\quad f'(0) = \frac{1}{4\cdot 2^{3/4}} = \frac{1}{2^{11/4}}$

$f''(x) = \frac{-3}{16}(x+2)^{-7/4}$ $\quad f''(0) = \frac{-3}{2^4\cdot 2^{7/4}} = \frac{-3}{2^{23/4}}$

$f'''(x) = \frac{21}{64}(x+2)^{-11/4}$ $\quad f'''(0) = \frac{21}{2^6\cdot 2^{11/4}} = \frac{21}{2^{35/4}}$

so Maclaurin series is $2^{1/4} + \frac{x}{2^{11/4}} - \frac{3}{2^{23/4}\cdot 2!}x^2 + \frac{21}{3!\,2^{35/4}}x^3 + \cdots$

could simplify a little: $2^{1/4} + \frac{x}{2^{11/4}} - \frac{3}{2^{27/4}}x^2 + \frac{7}{2^{39/4}}x^3 + \cdots$

Example 11 use same work from above

$f(14) = (14+2)^{1/4} = \sqrt[4]{16} = 2$

$f'(14) = \frac{1}{4}2^{-3} = \frac{1}{32}$ (using the fact that $16^{1/4} = 2$)

$f''(14) = -\frac{3}{16}\cdot 2^{-7} = \frac{-3}{2^{11}}$

$f'''(14) = \frac{21}{64}\cdot 2^{-11} = \frac{21}{2^{17}}$

so get

$$2 + \frac{1}{32}(x-14) - \frac{3}{2^{11}\cdot 2}(x-14)^2 + \frac{21}{2^{17}\cdot 3!}(x-14)$$

$$= 2 + \frac{1}{32}(x-14) - \frac{3}{2^{12}}(x-14)^2 + \frac{7}{2^{18}}(x-14)^3 + \cdots$$

Answers continued

Example 12 since $f(x) = \cos x$, $f'(x) = -\sin x$, etc. best point near $x = 15$ would be a multiple of π, such as 5π

Example 13 $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ so

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$$
$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

and $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

so $\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$

and so $\sin(x^2) + \cos(x^2) = \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) + \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right)$

$$= 1 + x^2 - \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Example 14 hah! trick question - a polynomial in x is its own Maclaurin Series! (find $f(x)$, $f'(x)$, $f''(x)$, etc. if you're not convinced!)

Example 15 $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

so $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \cdots$

the first term which has absolute value $< .05 = \frac{1}{20}$

is, well, not 1

not $\frac{1}{3!} = \frac{1}{6}$

but yes to $\frac{1}{5!} = \frac{1}{120} < \frac{1}{20}$

so using $1 - \frac{1}{3!}$ the first two non-zero terms gives the accuracy requested