

The Integral Test for the Convergence of a Series

Let $\sum_{k=1}^{\infty} a_k$ be a series such that $a_k = f(k)$ for $k = 1, 2, 3, \dots$ where the function f is positive, continuous, and decreasing on $[1, \infty)$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge.

Proof

Given the hypotheses of the theorem, we can illustrate the relationship between the series and the integral in two ways, shown in Figure H.1 (with circumscribed rectangles) and Figure H.2 (with inscribed rectangles).

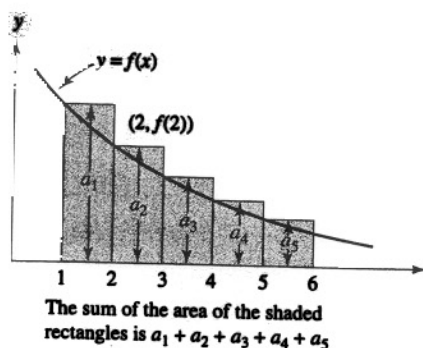


Figure H.1

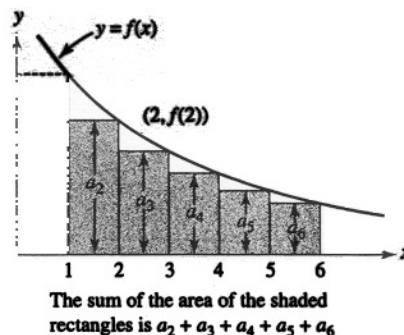


Figure H.2

Each rectangle has a base of 1, so its area corresponds to its height. In Figure H.1 the k th shaded rectangle has area $= a_k$; in Figure H.2 the k th shaded rectangle has area $= a_{k+1}$. Figure H.2 can be obtained from Figure H.1 by shifting the rectangles left one unit and dropping the rectangle corresponding to a_1 , from consideration.

$$a_2 + a_3 + \dots + a_n \leq \int_1^n f(x) dx \leq a_1 + a_2 + \dots + a_{n-1} \quad (\text{H.1})$$

We begin by demonstrating that the behavior of the series can be determined from that of the integral. The terms of the series are positive; therefore the sequence of partial sums is increasing.

- Suppose $\int_1^{\infty} f(x) dx$ converges. Denote $\int_1^{\infty} f(x) dx$ by A .

$$S_n = a_1 + a_2 + a_3 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

$$\lim_{n \rightarrow \infty} S_n \leq \lim_{n \rightarrow \infty} a_1 + \int_1^n f(x) dx = a_1 + A$$

The sequence of partial sums is bounded and increasing. Therefore, by the Bounded Increasing Partial Sums Theorem, $\sum_{k=1}^{\infty} a_k$ converges.

- Suppose $\int_1^{\infty} f(x) dx = \infty$. Refer to Equation (H.1) to obtain

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + \dots + a_n = S_n$$

Taking the limit as $n \rightarrow \infty$ gives

$$\infty \leq \lim_{n \rightarrow \infty} S_n.$$

So S_n grows without bound and $\sum_{k=1}^{\infty} a_k$ diverges.

Now we show that the behavior of the integral can be determined by that of the series.

Because $f(x) > 0$, decreasing, and continuous on $[1, \infty)$, $\lim_{b \rightarrow \infty} \int_1^b f(x) dx$ is either finite or grows without bound. Therefore, if we can find an upper bound, the integral converges. If it has no upper bound, it diverges.