Final Review (I)

(I) Integration

(1) $\int x^3 \cos(x^4+2)\,dx$

(2) $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

(3) $\int \sqrt{1+x^2}\; x^5\,dx$

(4) $\int \tan x\,dx$

(5) $\int x \sin x\,dx$

(6) $\int \ln x\,dx$

(7) $\int x^2 e^x\,dx$

(8) $\int e^x \sin x\,dx$

(9) $\int_0^1 \tan^{-1} x\,dx$

(10) $\int \cos^3 x\,dx$

(11) $\int \sin^5 x \cos^2 x\,dx$

(12) $\int \sin^4 x\,dx$

(13) $\int \tan^6 x \sec^4 x\,dx$

(14) $\int \tan^5 x \sec^7 x\,dx$

(15) $\int \sec x\,dx$

(16) $\int \tan^3 x\,dx$

(17) $\int \sec^3 x\,dx$

(18) $\int \sin 4x \cos 5x\,dx$

(19) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

(20) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

(21) $\int \frac{dx}{\sqrt{x^2-a^2}}$

(22) $\int \frac{x^3+x}{x-1}\,dx$

(23) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

(24) $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

(25) $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$

(26) $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

(27) $\int \frac{\sqrt{x+4}}{x}\, dx$

(31) $\int_2^5 \frac{1}{\sqrt{x-2}}\, dx$

(28) $\int_1^\infty \frac{1}{x}\, dx$

(32) $\int_0^{\frac{\pi}{2}} \sec x\, dx$

(29) $\int_{-\infty}^{0} x e^x\, dx$

(33) $\int_0^3 \frac{dx}{x-1}$

(30) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

(34) $\int_0^1 \ln x\, dx$

(35) For what value of p is the integral $\int_1^\infty \frac{1}{x^p}\, dx$ convergent ?

(36) Show that $\int_0^\infty e^{-x^2}\, dx$ is convergent by the comparision theorem.

(37) Show that $\int_1^\infty \frac{1+e^{-x}}{x}\, dx$ is divergent by the comparision theorem

(38) Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n=5$ to approximate the integral $\int_1^2 \frac{1}{x}\, dx$

(39) Use Simpson's Rule with $n=10$ to approximate $\int_1^2 \frac{1}{x}\, dx$.

Solution:

(I) Integration

(1) $\frac{1}{4}\sin(x^4+2)+C$ (5.5)

(2) $-\frac{1}{4}\sqrt{1-4x^2}+C$ (5.5)

(3) $\frac{1}{7}(1+x^2)^{7/2}-\frac{2}{5}(1+x^2)^{5/2}+\frac{1}{3}(1+x^2)^{3/2}+C$ (5.5)

(4) $\ln|\sec x|+C$ (5.5)

(5) $-x\cos x+\sin x+C$ (7.1)

(6) $x\ln x-x+C$ (7.1)

(7) $x^2e^x-2xe^x+2e^x+C$ (7.1)

(8) $\frac{1}{2}e^x(\sin x-\cos x)+C$ (7.1)

(9) $\frac{\pi}{4}-\frac{\ln 2}{2}$ (7.1)

(10) $\sin x-\frac{1}{3}\sin^3 x+C$ (7.2)

(11) $-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+C$ (7.2)

(12) $\frac{1}{4}(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x)+C$ (7.2)

(14) $\frac{1}{11}\sec^{11}x - \frac{2}{9}\sec^{9}x + \frac{1}{7}\sec^{7}x + C$ (7.2)

(15) $\ln|\sec x + \tan x| + C$ (7.2)

(16) $\frac{\tan^2 x}{2} - \ln|\sec x| + C$ (7.2)

(17) $\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$ (7.2)

(18) $\frac{1}{2}(\cos x - \frac{1}{9}\cos 9x) + C$ (7.2)

(19) $-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(\frac{x}{3}) + C$ (7.3)

(20) $-\frac{\sqrt{x^2+4}}{4x} + C$ (7.3)

(21) $\ln|x + \sqrt{x^2-a^2}| + C$ (7.3)

(22) $\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$ (7.4)

(23) $\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$ (7.4)

(24) $\frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$ (7.4)

(25) $x + \frac{1}{8}\ln(4x^2-4x+3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$ (7.4)

(26) $\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$ (7.4)

(27) $2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$ (7.4)

(28) dvg (7.8)

(29) -1 (7.8)

(30) π (7.8)

(31) $2\sqrt{3}$ (7.8)

(32) dvg (7.8)

(33) dvg (7.8)

(34) -1 (7.8)

(35) $p > 1$ (7.8)

(36) (37) (7.8)

(38) (a) ≈ 0.695635 (7.7)

(b) ≈ 0.691908

(39) ≈ 0.693150 (7.7)

Final Review (I)

(II) Series

(1) $\lim_{n\to\infty} \frac{n!}{n^n}$

(2) Use the Monotonic Sequence Theorem to prove $\lim_{n\to\infty} a_n$ exists and try to find what it is. Here the sequence $\{a_n\}$ is defined by the recurrence relation $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, \ldots$

(3) Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

(4) Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent ?

(5) Determine if the following series are absolutely convergent, conditionally convergent or divergent

(a) $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(c) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

(d) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

(e) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(g) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

(h) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

(i) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

(j) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

(k) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(l) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

(6) Find the interval of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} n!\, x^n$

(b) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

(d) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(e) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

(7) Find a power series representation of $\frac{1}{x+2}$

(8) Find a power series representation of $\frac{1}{(1-x)^2}$.

(hint: $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$)

(9) Evaluate $\int \left(\frac{1}{1+x^7}\right) dx$ as a power series

(10) Find the Taylor series for $f(x) = e^x$ at $a=2$

(11) Find the Maclaurin series for $f(x) = x\cos x$

(12) $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

(13) Find the 1st three nonzero terms in the Maclaurin series for

(a) $e^x \sin x$ (b) $\tan x$

(14) Expand $\frac{1}{(1+x)^2}$ as a power series

(15) Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of converge

(16) (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$

(b) How accurate is this approximation when $7 \le x \le 9$?

(17) (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \le x \le 0.3$? Using this approximation to find $\sin 12^\circ$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.0000

Solution:

(II) Series

(1) 0 (11.1)

(2) $\lim_{n\to\infty} a_n = 6$ (11.1)

(3) 3

(4) No

(5) ab cvg : (c) (d) (i) (j) (l)

cond cvg : (f) (h)

dvg : (a) (b) (e) (g) (k)

(11.2 ~ 11.6)

(6) (a) $\{0\}$ (11.9)

(b) $[2, 4)$

(c) $(-\infty, \infty)$

(d) $(-\frac{1}{3}, \frac{1}{3}]$

(e) $(-5, 1)$

(7) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$ (11.9)

(8) $\sum_{n=0}^{\infty} (n+1) x^n$

(9) $C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \cdots = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$

(10) $\sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$

(11) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$

(12) $\frac{1}{2}$

(13) (a) $x + x^2 + \frac{1}{3}x^3 + \cdots$

(b) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

(14) $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

(15) $f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n$; $R = 4$

(16) (a) $\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

(b) 0.0004

(17) (a) error $\approx 4.3 \times 10^{-8}$

$\sin 12^\circ \approx 0.2079116 9$

(b) $|x| < 0.82$

Final Review (I)

(III) Differential Equation

(1) (a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$

(b) Use (a) to sketch the solution curve that passes through the origin.

(2) Use Euler's method with step size 0.1 to approximate the value for $y(0.3)$ for the solution of the initial-value problem

$y' = x + y, \quad y(0) = 1$

(3) Solve $dy/dx = \dfrac{6x^2}{2y + \cos y}$ with $y(1) = \pi$

(4) Solve $y' = x^2 y$

(5) Find the orthogonal trajectories of the family of curves $x = ky^2$, whe

k is an arbitrary constant

(6) A tank contains 20 kg of salt dissolved in 5000 L of water. Brin

that contains 0.03 kg of salt per liter of water enters the tank at a

rate of 25 L/min. The solution is kept thoroughly mixed and drains fr

the tank at the same rate. How much salt remains in the tank after

half an hour?

(7) The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for

the mass of it that remains after t years.

(b) Find the mass after 1000 years.

(c) When will the mass be reduce to 30 mg?

(8) Write the solution of the initial-value problem $\frac{dP}{dt} = 0.08P(1 - \frac{P}{1000})$.
with $P(0) = 100$ and use it to find the population size $P(40)$ and $P(80)$
At what time does the population reach 900 ?

(9) Solve the differential equation $dy/dx + 3x^2y = 6x^2$.

(10) Find the solution of the differential equation $x^2y' + xy = 1$, $x > 0$
and $y(1) = 2$.

(11) Solve $y' + 2xy = 1$

(12) Suppose the populations of rabbits and wolves are discribed by the
Lotka-Volterra equation (i.e. predator-prey system) with $k = 0.08$, $a = 0$
$r = 0.02$ and $b = 0.00002$.
(a) Find the constant solutions (i.e. equilibrium solutions) and interpret
the answer.
(b) Find dW/dR
(c) Draw a direction field for dW/dR in the RW-plane. Then use
it to sketch some solution curves.
(d) Suppose that, at some point in time, there are 1000 rabbits and 4
wolves. Draw the corresponding solution curve and use it to describe
the changes in both population levels.

(13) Solve $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

(14) Solve $4y'' + 12y' + 9y = 0$

(15) Solve $y'' + y = 0$, $y(0) = 2$, $y'(0) = 3$

(16) Solve $y'' + y' - 2y = x^2$

(17) Solve $y'' + 4y = e^{3x}$

(18) Solve $y'' + y - 2y = \sin x$

(19) Solve $y'' - 4y = xe^{x} + \cos 2x$

(20) Solve $y'' + y = \sin x$

(21) A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

(22) Suppose that the spring of (21) is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any tim if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

(23) Use power series to solve $y'' + y = 0$

(24) (a) Solve $y'' - 2xy' + y = 0$ by using power series.

(b) Suppose $y(0) = 0$ and $y'(0) = 1$, find y.

Solution:

(III) Differential Equation

(1) see p 588 (9.2)

(2) 1.362 (9.2)

(3) $y^2 + \sin y = 2x^3 + \pi^2 - 2$ (9.3)

(4) $y = Ce^{x^3/3}$ (9.3)

(5) $x^2 + \frac{y^2}{2} = C$ (9.3)

(6) $150 - 130e^{-30/200}$ (9.3)

(7) (a) $100e^{-(\ln 2/1590)t} = 100 \times 2^{\frac{-t}{1590}}$ (9.4)

(b) $100e^{-(\ln 2/1590)\cdot 1000} \approx 65$

(c) $-1590\frac{\ln 0.3}{\ln 2} \approx 2762$

(8) $P(40) = \frac{1000}{1+9e^{-3.2}}$ (9.5)

$P(80) = \frac{1000}{1+9e^{-6.4}}$

$t = \frac{\ln 81}{0.08}$

(9) $y = 2 + Ce^{-x^3}$ (9.6)

(10) $y = \frac{\ln x + 2}{x}$ (9.6)

(11) $e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}$ (9.6)

(12) (a) $R=0$ & $W=0$ or $R=1000$ & $W=80$ (9.7)

(b) $$dW/dR = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(13) $$y = C_1 e^{(-1+\sqrt{13})x/6} + C_2 e^{(-1-\sqrt{13})x/6}$$ (17.1)

(14) $$y = C_1 e^{-3x/2} + C_2 x e^{-3x/2}$$ (17.1)

(15) $$y = 2\cos x + 3\sin x$$ (17.1)

(16) $$-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} + C_1 e^x + C_2 e^{-2x}$$ (17.2)

(17) $$C_1 \cos 2x + C_2 \sin 2x + \frac{1}{13} e^{3x}$$ (17.2)

(18) $$C_1 e^x + C_2 e^{-2x} - \frac{1}{10}(\cos x + 3\sin x)$$ (17.2)

(19) $$C_1 e^{2x} + C_2 e^{-2x} - \left(\frac{1}{3}x + \frac{2}{9}\right)e^x - \frac{1}{8}\cos 2x$$ (17.2)

(20) $$C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x$$ (17.2)

(21) $$\frac{1}{5}\cos 8t$$ (17.3)

(22) $$0.05\left(e^{-4t} - e^{-16t}\right)$$ (17.3)

(23) $$C_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + C_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ (17.4)

(24) (a) $$y = C_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \cdots\right) + C_1\left(x + \frac{1}{3!}x^3 + \cdots\right)$$ (17.4)

(b) $$y = x + \frac{1}{3!}x^3 + \frac{1.5}{5!}x^5 + \cdots$$