Series — Math 1b

Deinfition of convergence:
Let $\sum_{k=1}^{\infty} u_k$ be a series and $s_n = \sum_{k=1}^{n} u_k$ be the **n-th partial sum**. If $S = \lim_{n\to\infty} s_n$ exists, then $\sum_{k=1}^{\infty} u_k$ is said to **converge** with **sum** S. Otherwise, $\sum_{k=1}^{\infty} u_k$ is said to **diverge**.

Some classic series:

1. **Geometric series (GS)**: of the form $\sum_{k=1}^{\infty} ar^k$, where $a \neq 0$.
   - Converges to $a/(1-r)$ when $|r| < 1$. If $|r| \geq 1$ it diverges..
2. **p-series (PS)**: of the form $\sum_{k=1}^{\infty} 1/k^p$, where $p > 0$.
   - Converges if and only if $p > 1$.
3. **Alternating p-series**: of the form $\sum_{k=1}^{\infty}(-1)^k/k^p$.
   - Converges for all $p > 0$.

Types of series:

1. *arbitrary*: $\sum u_k$ with some terms possibly negative.
2. *positive*: $\sum a_k$ and $\sum b_k$ with $0 < a_l \leq b_l$ for sufficiently large l.
3. *alternating*: $\sum(-1)^k c_k$ with each $c_k > 0$.

Convergence tests:

1. **Divergence (DT)**:
   - If $\lim_{k\to\infty} \neq 0$ then $\sum u_k$ diverges.
   - If $\lim_{k\to\infty} = 0$ then use another test.
2. **Alternating Series (AST)**: $\sum(-1)^k c_k$ converges if the following hold:
   - $c_l > c_{l+1}$ for all sufficiently large l.
   - $\lim_{k\to\infty} c_k = 0$.

   To check whether this is conditional (or absolute) convergence, examine $\sum_{k=1}^{\infty} c_k$.
3. **Absolute Value (AVT)**: If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
4. **Ratio (RaT)**: Let $L = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$.
   - If $L < 1$ then $\sum_{k=1}^{\infty} u_k$ converges.
   - If $L > 1$ then $\sum_{k=1}^{\infty} u_k$ diverges.
   - If $L = 1$ then use another test.

5. **Root (RoT)**: Let $L = \lim_{k\to\infty} (|u_k|)^{\frac{1}{k}}$.
    - If $L < 1$ then $\sum_{k=1}^{\infty} u_k$ converges.
    - If $L > 1$ then $\sum_{k=1}^{\infty} u_k$ diverges.
    - If $L = 1$ then use another test.
6. **Comparison (CT)**:
    - If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
    - If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
7. **Limit Comparison (LCT)**: Let $L = \lim_{k\to\infty} \frac{a_k}{b_k}$. If $0 < L < \infty$, then
    - $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, or
    - $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge.
8. **Integral (IT)**: Let $f(x)$ be a continuous, positive, decreasing function with $f(l) = a_l$ for l sufficiently large. Then
    - $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_{k=1}^{\infty} f(x)\, dx$ converges.

Summary of techniques:
The following table tells us which of the above tests can be used for particular classes of series. The final column gives a tip as to when each test is often useful.

	Positive	Alternating	Arbitrary	When to use
DT	$\surd$	$\surd$	$\surd$	Terms get “big”
AST		$\surd$		Have alternating series
AVT		$\surd$	$\surd$	“Positive” form converges
RaT	$\surd$	$\surd$	$\surd$	factorials, k-th powers
RoT	$\surd$	$\surd$	$\surd$	k-th powers
CT	$\surd$			Similar to PS or GS
LCT	$\surd$			Similar to PS or GS
IT	$\surd$			Easy to integrate

Some questions to consider:

1. Give examples of series $\sum_{k=1}^{\infty} a_k$ for which $L = 1$ in the ratio test.
2. Suppose $\sum_{k=1}^{\infty} a_k$ is a series and $|a_{k+1}|/|a_k| = 1/6$ for every k. Can I make any conclusions about convergence?
3. Why does the AST let you conclude convergence in some cases whereas the DT never lets you conclude convergence?
4. Does convergence of $\sum_{k=1}^{\infty} a_k$ tell you anything about convergence of $\sum_{k=6}^{\infty} a_k$? What about their respective sums?
5. Calculate $\lim_{x\to\infty} x \cdot \tan(1/x)$.