INTEGRATION TECHNIQUES
MATH1B — FALL 2000

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1. Integrals I've known

You will be expected to know all of the integrals in the following table. Note that the ones in the second column are *very* similar to the integrals appearing in the left column. I have left out the constants of integration.

$\int \cos(x)dx = \sin(x)$	$\int \sin(x)dx = -\cos(x)$
$\int \sec^2(x)dx = \tan(x)$	$\int \csc^2(x)dx = -\cot(x)$
$\int \sec(x)\tan(x)dx = \sec(x)$	$\int \csc(x)\cot(x)dx = -\csc(x)$
$\int \sec(x)dx = \ln\lvert\sec(x)+\tan(x)\rvert$	$\int \csc(x)dx = \ln\lvert\csc(x)-\cot(x)\rvert$
$\int \tan(x)dx = \ln\lvert\sec(x)\rvert$	$\int \cot(x)dx = -\ln\lvert\csc(x)\rvert$
$\int \frac{dx}{1+x^2} = \tan^{-1}(x)$	$\int \frac{-dx}{1+x^2} = \cot^{-1}(x)$
$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)$	$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1}(x)$
$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x)$	$\int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}(x)$

2. Substitution

When to use:

- Your integrand has the composition of two functions and "some other stuff."
- Your integrand consists of the product of "similar" looking things.

How to do it:
If $u = g(x)$, then $du = g'(x)\ dx$ and $\int f(g(x))g'(x)\ dx = \int f(u)\ du$.

Potential pitfalls:

- When using substitution for *definite* integrals, remember to either:
  - Switch back to original variable before evaluating at end-points. Or
  - Rewrite limits of integration in terms of u.
- Don't forget to replace dx.

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3. Integration by parts

When to use:

- Your integrand has the product of "dissimilar" things. For example,
  - $f(x)e^{g(x)}$ where $\deg(f(x)) \neq \deg(g(x)) - 1$ (here $f(x),\ g(x)$ are polynomials).
  - $\ln(x)f(x)$ where $f(x)$ is a polynomial
  - $e^{f(x)}t(x)$, where $t(x)$ is some trig function.
  - $p(x)t(x)$ where $p(x)$ is a polynomial and $t(x)$ is a trig function.

How to do it:
Integration by parts consists of reversing the product rule for differentiation. In particular, you apply either of the two following (equivalent) formulas:

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Or, setting $u = f(x)$ and $dv = g'(x)dx$, we can rewrite this as

$$\int_a^b u\ dv = uv|_a^b - \int_a^b v\ du$$

Tips:

- Choose dv to be as complicated as possible so that you still know how to integrate it easily.
- Choose u so that du is simpler than u itself.
- Try to balance the previous two criteria.
- Keep an eye out for looping integrals when using this technique (especially if your integrand contains exponential or trig functions).
- If your integrand doesn't have any quotients, set u to be the first type of function appearing in the following list: *Logarithm*, *Inverse trig function*, *Polynomial*, *Exponential*, *Trig. function*. The list can be remembered by the acronym LIPET.

4. Trig. substitution

When to use:

- You have $\sqrt{\ }$'s in your integrand.

How to do it:

- To get rid of $\sqrt{x^2 - a^2}$, set $x = a\sec(\theta)$.
- To get rid of $\sqrt{a^2 - x^2}$, set $x = a\sin(\theta)$.
- To get rid of $\sqrt{a^2 + x^2}$, set $x = a\tan(\theta)$.

Pitfalls:

- Make sure to distinguish between the first two cases above.
- Don't forget to replace dx.
- You may need to "complete the square" before you will be able to use one of the above substitutions.

5. Trig. integrals

When to use:

- Your integrand is of the form $\sin^m(x)\cos^p(x)$.
- Your integrand is of the form $\sec^m(x)\tan^p(x)$.

How to do it:

If:	save a copy of:	convert the rest to:	using:
m is odd	$\sin(x)$	$\cos(x)$	$\sin^2(x) = 1 - \cos^2(x)$
p is odd	$\cos(x)$	$\sin(x)$	$\cos^2(x) = 1 - \sin^2(x)$
p is odd	$\sec(x)\tan(x)$	$\sec(x)$	$\tan^2(x) = \sec^2(x) - 1$
m is even	$\sec^2(x)$	$\tan(x)$	$\sec^2(x) = 1 + \tan^2(x)$

Pitfalls:

- If $\sin(x)$ and $\cos(x)$ both appear to even powers, use the half-angle identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$. These identities (if needed) will be given to you on the exam.
- If $\sec(x)$ appears to an odd power and $\tan(x)$ to an even power, then integration by parts is probably your best bet.

6. Improper integrals

When to use it:

- You have a definite integral on an interval for which the function has a vertical asymptote.
- One of your limits of integration is $\pm\infty$.

How to do it:

- If you're evaluating $\int_a^\infty f(x)dx$, first calculate $\int_a^b f(x)dx$. Then take $\lim_{b\to\infty}$ of your answer.
- If you're evaluating $\int_a^b f(x)dx$ and $f(x)$ has a vertical asymptote at c with $a \le c \le b$, then calculate both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$.

Potential pitfalls:

- Always use limits when evaluating improper integrals. If you just plug in when not appropriate you will lose points. Worse, you may get the wrong answer.
- If you split up the integral due to an asymptote, you need *both* of your integrals to converge (i.e., be finite) in order to say that the original integral converges.

7. Partial fractions

When to use:

- You have an integral of the form $\int \frac{f(x)}{g(x)} dx$, where $f(x)$ and $g(x)$ are both polynomials.

Potential pitfalls:

- If $\deg(f(x)) \geq \deg(g(x))$, use long division.
- If $f(x) = g'(x)$, the integral is just $\ln |g(x)|$.
- If this is a definite integral, make sure it isn't really an improper integral.

How to do it:

- Factor the denominator as much as you can:
  - You will not be able to factor the polynomial $ax^2 + bx + c$ if $b^2 - 4ac < 0$.
  - If you have a cubic polynomial in the denominator, you can probably factor out an x.
- Write your integrand as the sum of other fractions:
  - For each $(ax+b)^r$ in your denominator, include the following fractions in your sum.
  $$\frac{A_1}{ax+b} + \cdots + \frac{A_r}{(ax+b)^r}$$
  - For each $(ax^2+bx+c)^r$ in your denominator (that can't be factored), include the following fractions in your sum.
  $$\frac{B_1 x + C_1}{ax^2+bx+c} + \cdots + \frac{B_r x + C_r}{(ax^2+bx+c)^r}$$
- Solve for the constants A_i, B_i, and C_i by
  - plugging in nice values of x
  - collecting coefficients of equal powers of x.
- Check that you found the partial fractions correctly.
- Integrate your partial fractions.

8. Other points

Some points/questions to ruminate on:

- The best way to learn this material is to do *lots* of integrals for which you are not told in advance the appropriate technique.
- Always check to make sure your integral isn't an easy one before hitting it with a powerful technique.
- What is
$$\int \frac{x\ dx}{\sqrt{x^2-9}}\ dx$$
- Does $\int_{-2}^{2} dx/x$ converge?
- Which of the following two equivalent integrals is easier?
$$\int \sec^2(x)\ dx \text{ or } \int (\tan^2(x)+1)\ dx$$
- If $\cot(\theta) = (\sqrt{x-2})/3$, what is an expression for $\sin(\theta)$ in terms of x?