SOLUTIONS:

§9.5 THE LOGISTIC EQUATION

$\boxed{3}$ $\frac{dy}{dt} = ky\left(1-\frac{y}{K}\right)$

$$\frac{dy}{y(1-y/K)} = k\,dt$$

$$\int\left(\frac{1}{y}+\frac{1}{K-y}\right)dy = \int k\,dt$$

$$\ln|y| - \ln|K-y| = kt + C$$

$$\ln\left|\frac{y}{K-y}\right| = kt + C$$

$$\frac{y}{K-y} = e^{kt+C} = e^{C}e^{kt}$$

$$\frac{K-y}{y} = \underbrace{e^{-C}}_{A}e^{-kt}$$

$$\frac{K}{y} - 1 = Ae^{-kt}$$

$$\frac{K}{y} = 1 + Ae^{-kt}$$

$$\boxed{\frac{K}{1+Ae^{-kt}} = y}$$

(a) $K = 8\times10^7$ kg, $k = .71/\text{yr}$

$$\frac{K}{y(0)} - 1 = Ae^{0}$$

$$\frac{8\times10^7\,\text{kg}}{2\times10^7\,\text{kg}} - 1 = A \rightarrow \boxed{A=3}$$

So $\boxed{y = \frac{8\times10^7}{1+3e^{-.71t}}}$

$$y(1) = \frac{8\times10^7}{1+3e^{-.71}} \approx 3.23\times10^7\,\text{kg}$$

(b) Set $\frac{8\times10^7}{1+3e^{-.71t}} = 4\times10^7$

$$2 = 1+3e^{-.71t}$$

$$1/3 = e^{-.71t}$$

$$\ln(1/3) = -.71t$$

$$t = \frac{\ln 3}{.71} \approx 1.55\,\text{yrs}$$

$\boxed{6}$ Let's assume world carrying capacity is ~100 billion, in which case US carrying capacity is ~ 3-5 billion. I'll choose US carrying capacity ≡ 4 billion = 4000 million.

In millions: $\boxed{P(t) = \frac{4000}{1+Ae^{-kt}}}$

(a) Let $t=0$ be 1980. Then $228 = \frac{4000}{1+A} \rightarrow 1+A = \frac{4000}{228} \rightarrow A = \frac{4000}{228} - 1 = \frac{943}{57}$

$$\boxed{P(t) = \frac{4000}{1+\frac{943}{57}e^{-kt}}}$$

Now solve for k:

$$P(10) = 250 = \frac{4000}{1+\frac{943}{57}e^{-10k}} \rightarrow 1+\frac{943}{57}e^{-10k} = 16 \rightarrow e^{-10k} = \frac{855}{943}$$

$$-10k = \ln\frac{855}{943} \rightarrow \boxed{k = -\frac{1}{10}\ln\frac{855}{943} \approx .0097965}$$

6 (c) 2100: $P(120) = \frac{4000}{1+\frac{943}{57}e^{-120(.0098)}} \approx 655$ million

2200: $P(220) = \frac{4000}{1+\frac{943}{57}e^{-220(.0098)}} \approx 1371$ million $= 1.4$ billion

(d) Set $\frac{4000}{1+\frac{943}{57}e^{-kt}} = 300 \rightarrow 1 + \frac{943}{47}e^{-kt} = \frac{40}{3} \rightarrow e^{-kt} = \frac{37}{3}\cdot\frac{57}{943}$

$\rightarrow -kt = \ln\frac{703}{943} \rightarrow t = \frac{10\ln\frac{703}{943}}{\ln\frac{855}{943}} \approx 29.98 = 30$

So US population will exceed 300 million in the year 2010

optional: (7) (a) Assume $\frac{dy}{dt} = ky(1-y)$, where y is fraction of population having heard rumor

(b) Put this eqn in form $\frac{dP}{dt} = kP(1-\frac{P}{K})$, so we can solve by inspection, or,

$\frac{dy}{y(1-y)} = k\,dt \rightsquigarrow \int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int k\,dt \rightsquigarrow \ln|y| - \ln|1-y| = kt + C$

$\rightsquigarrow \frac{y}{1-y} = e^{c}e^{kt} \rightarrow \frac{1}{y} - 1 = \underbrace{e^{-c}}_{A}e^{-kt} \rightsquigarrow y = \frac{1}{1+Ae^{-kt}}$

$y(0) = \frac{1}{1+A} \rightsquigarrow 1+A = \frac{1}{y_0} \rightarrow A = \frac{1}{y_0} - 1 \rightsquigarrow y = \frac{1}{1+(\frac{1}{y_0}-1)e^{-kt}}$

$$\boxed{y = \frac{y_0}{y_0 + (1-y_0)e^{-kt}}}$$

(c) let $t=0$ be 8 AM. So $y_0 = \frac{80}{1000} = .08$, $y(4) = \frac{1}{2}$

So $\frac{1}{2} = \frac{.08}{.08+(1-.08)e^{-4k}} \rightsquigarrow .08 + .92e^{-4k} = .16 \rightsquigarrow e^{-4k} = \frac{.08}{.92} = \frac{2}{23}$

So $e^{-k} = (2/23)^{1/4}$

$$\boxed{y(t) = \frac{.08}{.08 + .92\,(2/23)^{t/4}} = \frac{2}{2+23(2/23)^{t/4}}}$$

Set $.90 = \frac{2}{2+23(2/23)^{t/4}} \rightsquigarrow 2 + 23\left(\frac{2}{23}\right)^{t/4} = \frac{2}{.9} = \frac{20}{9} \rightsquigarrow 23\left(\frac{2}{23}\right)^{t/4} = \frac{2}{9}$

$\left(\frac{2}{23}\right)^{t/4} = \frac{2}{(9)(23)} \rightsquigarrow \frac{t}{4}\ln\left(\frac{2}{23}\right) = \ln\left(\frac{2}{(9)(23)}\right) \rightsquigarrow t = \frac{4\left(\ln\left(\frac{2}{(9)(23)}\right)\right)}{\ln\left(\frac{2}{23}\right)}$

$\boxed{t \approx 7.6 \approx 7\text{ hr }36\text{ min}}$. So $\boxed{90\% \text{ hears by } 3{:}36\text{ PM}}$

(8) (a) $P(0) = P_0 = 400$, $K = 10000$, $P(1) = 3\cdot 400 = 1200$

$P(t) = \frac{P_0K}{P_0 + (K-P_0)e^{-kt}}$, as before

$P = \frac{400(10000)}{400 + 9600e^{-kt}} = \frac{10000}{1+24e^{-kt}}$

② 8 a) $P(1)=1200 \Rightarrow 1+24e^{-k}=\frac{100}{12} \rightsquigarrow e^{k}=\frac{288}{88} \rightsquigarrow k=\ln\frac{36}{11}$

So $\boxed{P=\frac{10000}{1+24e^{-t\ln(36/11)}}=\frac{10000}{1+24\cdot(11/36)^{t}}}$

ⓑ Set $5000=\frac{10000}{1+24\left(\frac{11}{36}\right)^{t}} \rightarrow 24\left(\frac{11}{36}\right)^{t}=1 \rightarrow t\ln\left(\frac{11}{36}\right)=\ln\left(\frac{1}{24}\right)$

$\boxed{t=\ln(1/24)/\ln(11/36)\approx 2.68}$

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⑬ ⓐ $\frac{dP}{dt}=(kP)\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)$

If $m<P<K$, then $\frac{dP}{dt}=(+)(+)(+)=+ \rightsquigarrow P$ is increasing.

If $0<P<m$, then $\frac{dP}{dt}=(+)(+)(-)=- \rightsquigarrow P$ is decreasing

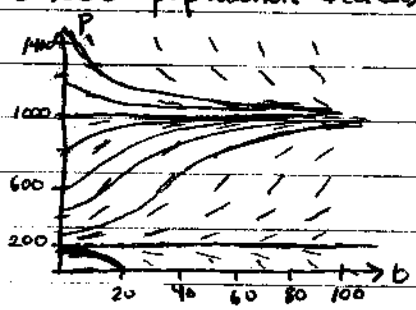
ⓑ $k=.08$, $K=1000$, $m=200$

$\frac{dP}{dt}=.08P(1-P/1000)(1-200/P)$

For $0<P_0<200$, population dies out. $P_0=200$: steady population

$200<P_0<1000$ population increases, approaching 1000. $P_0=1000$: steady

$P_0>1000$: population decreases, approaching 1000.



Equilibrium soln's:

$P(t)=200$, $P(t)=1000$

ⓒ $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)=\frac{k}{K}(K-P)(P-m)$

$\int\frac{dP}{(K-P)(P-m)}=\int\frac{k}{K}dt$

Partial fractions

$\frac{1}{(K-P)(P-m)}=\frac{A}{K-P}+\frac{B}{P-m}$

$1=A(P-m)+B(K-P)$

$P=m$: $1=B(K-m) \rightarrow B=\frac{1}{K-m}$

$P=K$: $1=A(K-m) \rightarrow A=\frac{1}{K-m}$

$\int\left(\frac{1/(K-m)}{K-P}+\frac{1/(K-m)}{P-m}\right)dP=\int\frac{k}{K}dt$

$\frac{1}{K-m}\int\left(\frac{1}{K-P}+\frac{1}{P-m}\right)dP=\int\frac{k}{K}dt$

$\frac{1}{K-m}\left[-\ln|K-P|+\ln|P-m|\right]=\frac{k}{K}t+C$

$\ln\left(\frac{P-m}{K-P}\right)=(K-m)\frac{k}{K}t+C_1 \rightarrow \frac{P-m}{K-P}=C_2e^{(K-m)\left(\frac{k}{K}\right)t}$

At $t=0$, $\frac{P_0-m}{K-P_0}=C_2$, so $\frac{P-m}{K-P}=\frac{P_0-m}{K-P_0}e^{(K-m)\left(\frac{k}{K}\right)t}$

Solving for P: $P-m=K\left(\frac{P_0-m}{K-P_0}\right)e^{(K-m)(k/K)t}-P\left(\frac{P_0-m}{K-P_0}\right)e^{(K-m)\left(\frac{k}{K}\right)t}$

$P\left(1+\frac{P_0-m}{K-P_0}e^{(K-m)(k/K)t}\right)=m+K\left(\frac{P_0-m}{K-P_0}\right)e^{(K-m)(k/K)t}$

13c) So $\boxed{P(t) = \dfrac{m(K-P_0) + K(P_0-m)e^{(K-m)(k/K)t}}{K-P_0 + (P_0-m)e^{(K-m)(k/K)t}}}$

d) If $P_0 < m$, then $P_0 - m < 0$.
Consider the numerator of $P(t)$. (call it $N(t)$).
$N(0) = m(K-P_0) + K(P_0-m) = P_0(K-m)$, which is >0.
So if $P_0 - m < 0$, then $\lim_{t\to\infty} K(P_0-m)e^{(K-m)(k/K)t} \to -\infty$, so $N(t) \to -\infty$, t large.

Since N is continuous and it starts out positive and becomes negative, it must pass through 0; ie there exists t such that $N(t) = 0$, and thus $P(t) = 0$. So at some point, $P=0$: *extinction*.

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14) a) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P$

$\displaystyle\int \frac{dP}{P\ln(K/P)} = \int ct$

let $u = \ln(K/P)$
$u = \ln K - \ln P$
$du = -dP/P$

$\displaystyle\int -\frac{du}{u} = ct + D$

$-\ln|u| = ct + D$

$|u| = e^{-(ct+D)}$

$|\ln(K/P)| = e^{-(ct+D)} = D_1 e^{-ct}$

$\ln(K/P) = D_1 e^{-ct}$ $(K \geq P)$

At $t=0$, $\ln(K/P_0) = D_1$

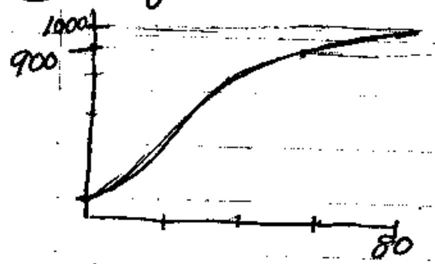
So $\ln(K/P) = \ln(K/P_0)e^{-ct}$

$K/P = e^{\ln(K/P_0)e^{-ct}}$

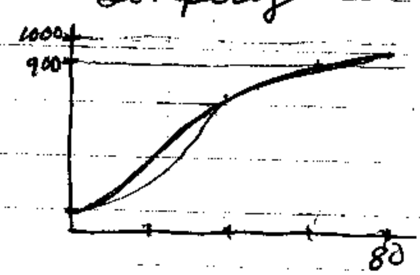
$\boxed{P(t) = Ke^{-\ln(K/P_0)e^{-ct}}}$

b) $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Ke^{-\ln(K/P_0)e^{-ct}} = Ke^{-\ln(K/P_0)\cdot 0} = Ke^0 = K$

c) Logistic



Gompertz



The graphs look very similar. For lower t, say $t=40$, both functions are ≈ 732, so functions are $\approx$ equal. For higher t, they diverge slightly. $P_{logistic}(55) \approx 900$, whereas $P_{Gompertz}$ doesn't reach 900 until $t=62$. Also $P_{logistic}(80) \approx 985$, while $P_{Gompertz}$ is only ≈ 960, so for higher t, $P_{Gompertz}$ does not increase as fast as $P_{logistic}$.

d) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P = cP(\ln K - \ln P)$

$\dfrac{d^2P}{dt^2} = c\left[P\left(-\dfrac{1}{P}\dfrac{dP}{dt}\right) + (\ln K - \ln P)\dfrac{dP}{dt}\right] = c\dfrac{dP}{dt}\left[-1 + \ln\left(\dfrac{K}{P}\right)\right]$

③ 14(b) $\frac{d^2P}{dt^2} = c\left(c\ln\left(\frac{K}{P}\right)P\right]\left[\ln\left(\frac{K}{P}\right)-1\right]$

$$P'' = c^2 P \ln(K/P)[\ln(K/P)-1]$$

For $0<P<K$, $P''=0 \leftrightarrow \ln(K/P)=1 \leftrightarrow K/P = e \leftrightarrow P = K/e$

$P''>0$ for $0<P<K/e$

$P''<0$ for $K/e<P<K$

So P' is maximum (P grows fastest) when $P = K/e$.

⑮ⓐ $\frac{dP}{dt} = kP\cos(rt-\phi) \rightarrow \int\frac{dP}{P} = \int k\cos(rt-\phi)\,dt$

optional

$\rightarrow \ln|P| = \frac{k}{r}\sin(rt-\phi)+C$

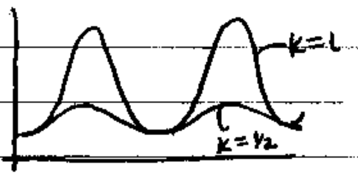
$P(0)=P_0 \rightarrow \ln P_0 = \frac{k}{r}\sin(-\phi)+C \rightarrow C = \ln P_0 + \frac{k}{r}\sin\phi$

So $\ln P = \frac{k}{r}\sin(rt-\phi) + \ln P_0 + \frac{k}{r}\sin\phi$

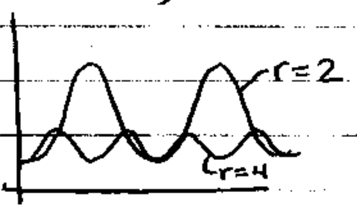
$\ln(P/P_0) = \frac{k}{r}[\sin(rt-\phi)+\sin\phi]$

$P(t) = P_0 e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$

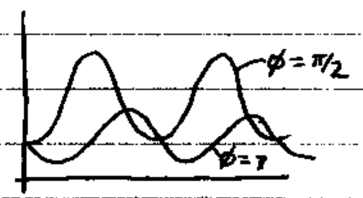
ⓑ As k increases amplitude increases but minimum value stays same:



As r increases, amplitude & period decrease:



A change in ϕ produces smaller adjustments in phase shift & amplitude:



$P(t)$ oscillates between $P_0 e^{(k/r)[1+\sin\phi]}$ and $P_0 e^{(k/r)[-1+\sin\phi]}$,

So $\lim_{t\to\infty} P(t)$ does not exist.