§ 9.4 : 4, 6, 10, 12, (13), 14, (19), 20

4a. $y(t) = y(0)e^{kt}$

$\rightarrow y(2) = y(0)e^{2k} = 400, \quad y(6) = y(0)e^{6k} = 25{,}600$

$\frac{y(6)}{y(2)} = \frac{e^{6k}}{e^{2k}} = e^{4k} = \frac{25{,}600}{400} = 164$

$\rightarrow k = \frac{3}{2}\ln 2 = \frac{1}{2}\ln 8$

So $y(0) = 400/e^{2k} = 400/e^{\ln 8} = 400/8 = 50$

b. $y(t) = y(0)e^{kt} = 50\, e^{(\frac{1}{2}\ln 8)t} = 50 \cdot 8^{t/2}$

c. $100 = 50 e^{(\frac{1}{2}\ln 8)t} \rightarrow 2 = e^{(\frac{1}{2}\ln 8)t} \rightarrow t = \frac{2}{3}$ hr

d. $100{,}000 = 50 e^{(\frac{1}{2}\ln 8)t} \rightarrow 2000 = e^{(\frac{1}{2}\ln 8)t} \rightarrow t \approx 7.3$ hr

6 a. Say $P(t)$ is the population in millions at time t. The initial time is 1900, so substitute $t-1900$ for t in Thm 2. $\rightarrow P(t) = P(1900)\,e^{k(t-1900)}$

$\rightarrow P(1910) = 92 = 76e^{k(1910-1900)}$

$\rightarrow k = \frac{1}{10}\ln\frac{92}{76} \approx 0.0191$

So estimate $P(1990) \approx 76e^{0.0191(90)} \approx 424$ million. This is very high. The reason is that the number of immigrants was very large between 1900 and 1910 compared to the actual population. Since then this ratio has decreased, as has the birth rate in the U.S., so the calculated k is based on out of date figures.

b. Substituting $t-1970$ for t in Thm 2 gives $P(t) = P(1970)\,e^{k(t-1970)}$

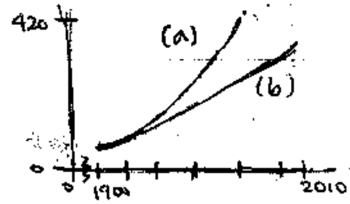
$\rightarrow P(1980) = 227 = 203e^{k(1980-1970)}$

$\rightarrow k = \frac{1}{10}\ln\frac{227}{203} \approx 0.01117$

So estimate $P(1990) \approx 203e^{0.01117(20)} \approx 254$ million. This is very close.

$P(2000) \approx 203\,e^{0.01117(30)} \approx 284$ million

$P(2010) \approx 203\,e^{0.01117(40)} \approx 317$ million

c. 

The model from (a) was not very accurate after 1910. The model from (b) was much better.

10a. Let $y(t)$ be the mass after t days, and $y(0)=A$. Then $y(t)=Ae^{kt}$.

$y(3) = Ae^{3k} = 0.58A \rightarrow e^{3k} = 0.58$

$\rightarrow k = \frac{1}{3}\ln 0.58$

Then $Ae^{(\ln .58)t/3} = .5A \rightarrow \frac{1}{3}(\ln .58)t = \ln .5 \rightarrow t \approx 3.82$ days. $= \frac{-3\ln 2}{\ln 0.58}$

b. $Ae^{(\ln .58)t/3} = .1A \rightarrow \frac{1}{3}(\ln .58)t = \ln .1 \rightarrow t \approx 12.68$ days $= -\frac{3\ln 10}{\ln 0.58}$

12. $\frac{dy}{dx} = 2y$. So $y = Ce^{2x}$ by Thm 2.

$y(0) = 5 = Ce^0 = C$. So the equation for the curve is $y=5e^{2x}$

13a. If $y = u - 75$, $u(0) = 185 \rightarrow y(0) = 185 - 75 = 110$.

So the initial value problem is $\frac{dy}{dt} = ky$ with $y(0) = 110$

$\rightarrow y(t) = 110e^{kt}$

b. $y(30) = 110e^{30k} = 150 - 75$

$\rightarrow e^{30k} = \frac{15}{22} \rightarrow k = \frac{1}{30}\ln\frac{15}{22}$. So $y(t) = 110e^{\frac{t}{30}\ln\frac{15}{22}}$

$y(45) = 110e^{\frac{45}{30}\ln(\frac{15}{22})} \approx 62°$

So $u(45) \approx 62 + 75 = 137°F$.

c. $u(t) = 100 \rightarrow y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\frac{15}{22}} = 25$

$\rightarrow t = \frac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

14a. Let $y(t)$ be the temperature after t minutes.
Newton's Law of Cooling says $\frac{dy}{dt} = k(y-5)$. Let $u(t) = y(t) - 5$.
Then $\frac{du}{dt} = ku$ so $u(t) = u(0)e^{kt} = 15e^{kt}$

$\rightarrow y(t) = 15e^{kt} + 5$.

$y(1) = 12 = 5 + 15e^{kt} \rightarrow k = \ln\frac{7}{15}$. So $y(t) = 15e^{(\ln\frac{7}{15})t} + 5$.

$y(2) = 5 + 15e^{2\ln\frac{7}{15}} \approx 8.3°\ C$

b. $5 + 15e^{(\ln\frac{7}{15})t} = 6$ when $e^{(\ln\frac{7}{15})t} = \frac{1}{15}$

$\rightarrow t = \frac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.6$ min.

19a. $\frac{dP}{dt} = kP - m = k\left(P - \frac{m}{k}\right)$. Let $y = P - \frac{m}{k}$.
So $\frac{dy}{dt} = ky$ and we have $y = y_0 e^{kt}$.

$\rightarrow P(t) = \frac{m}{k} + \left(P_0 - \frac{m}{k}\right)e^{kt}$

b. There will be an exponential expansion of the population exactly when $P_0 - \frac{m}{k} > 0$, so $m < kP_0$.

c. The population will be constant if $P_0 - \frac{m}{k} = 0$, so $m = kP_0$.
It will decline if $P_0 - \frac{m}{k} < 0$, so $m > kP_0$.

d. $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000$.
$m > kP_0$ so by part (c), the population was declining.

20a. $\frac{dy}{dt} = ky^{1+c} \rightarrow \int y^{-1-c}\,dy = \int k\,dt$

$\frac{y^{-c}}{-c} = kt + d$ (d a constant)

Since $y(0) = y_0$, $d = \frac{y_0^{-c}}{-c}$. So $y^{-c} = y_0^{-c} - ckt$

$\rightarrow y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^c kt} \rightarrow y(t) = \frac{y_0}{(1 - cy_0^c kt)^{1/c}}$

b. $y(t) \to \infty$ as $1 - cy_0^c kt \to 0$. That is, as $t \to \frac{1}{cy_0^c k}$

Let $T = \frac{1}{cy_0^c k}$. Then $\lim_{t \to T^-} y(t) = \infty$

c. So $c = 0.01$, $y(0) = 2$, and $y(3) = 16$ where t is in months.

Thus $y_0 = 2$ and $16 = y(3) = \frac{y_0}{(1 - cy_0^c k \cdot 3)^{1/c}}$

Since $T = \frac{1}{cy_0^c k}$, let's solve for $cy_0^c k$.

$$16 = \frac{2}{(1 - 3cy_0^c k)^{100}} \longrightarrow 1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01}$$

$$\longrightarrow cy_0^c k = \frac{1}{3}\left(1 - 8^{-0.01}\right)$$

Thus doomsday occurs when $t = T = \frac{1}{cy_0^c k} = \frac{3}{1 - 8^{-.01}} \approx 145.77$ months or 12.15 years.