§ 7.1 : 6, 10, 18, 28, 38, 48

6. $\int \sin^{-1}x\,dx$ Let $u = \sin^{-1}x$ and $dv = dx$ $\rightarrow du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$.

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

Set $t = 1-x^2$. Then $dt = -2x\,dx$, so $x\,dx = -\frac{1}{2}dt$

So $-\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{1}{2} t^{-1/2}\,dt = t^{1/2} + c = \sqrt{1-x^2} + c$.

$\rightarrow \int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + c$

10. $\int t^3 e^t\,dt$ Let $u_1 = t^3$ and $dv_1 = e^t\,dt$ $\rightarrow du_1 = 3t^2\,dt$ and $v_1 = e^t$.

$\int t^3 e^t\,dt = t^3 e^t - \int 3t^2 e^t\,dt$ Let $u_2 = 3t^2$ and $dv_2 = e^t\,dt$ $\rightarrow du_2 = 6t\,dt$ and $v_2 = e^t$

So $-\int 3t^2 e^t\,dt = -3t^2 e^t + \int 6te^t\,dt$ Let $u_3 = 6t$ and $dv_3 = e^t\,dt$ $\rightarrow du_3 = 6\,dt$ and $v_3 = e^t$

So $\int 6te^t\,dt = 6te^t - \int 6e^t\,dt = 6te^t - 6e^t + c$

$\rightarrow \int t^3 e^t\,dt = t^3e^t - 3t^2e^t + 6te^t - 6e^t + c$

$= e^t(t^3 - 3t^2 + 6t - 6) + c$

(In general, if $p(t)$ is a polynomial of degree n in t then repeated integration by parts shows that $\int p(t)e^t\,dt = [p(t) - p'(t) + p''(t) - \dots + (-1)^n p^{(n)}(t)]e^t + c$)

18. $\int_1^4 \sqrt{t}\ln t\,dt$ Let $u = \ln t$ and $dv = \sqrt{t}\,dt$. $\rightarrow du = \frac{dt}{t}$ and $v = \frac{2}{3}t^{3/2}$

By formula 6, $\int_1^4 \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{t}\,dt$

$= \frac{2}{3}\cdot 8\ln 4 - 0 - \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{28}{9}$

28. $\int_0^t e^s \sin(t-s)\,ds$. Let $u=\sin(t-s)$ and $dv=e^s ds$

$\rightarrow du=-\cos(t-s)ds$ and $v=e^s$

$\rightarrow \int_0^t e^s \sin(t-s)ds = \sin(t-s)e^s\Big|_0^t + \int_0^t \cos(t-s)e^s ds$

$= -\sin t + \int_0^t \cos(t-s)e^s ds$

Let $U=\cos(t-s)$ and $dV=e^s ds$. $\rightarrow dU=+\sin(t-s)ds$ and $V=e^s$

So $\int_0^t \cos(t-s)e^s ds = \cos(t-s)e^s\Big|_0^t - \int_0^t \sin(t-s)e^s ds$

$= e^t - \cos t - \int_0^t \sin(t-s)e^s ds$.

So $\int_0^t e^s \sin(t-s) = -\sin t + e^t - \cos t - \int_0^t \sin(t-s)e^s ds$.

$\longrightarrow \int_0^t e^s \sin(t-s) = \frac{1}{2}(e^t - \sin t - \cos t)$

38a. $\int \cos^n x\,dx$ Let $u=\cos^{n-1}x$ and $dv=\cos x\,dx$

$\longrightarrow du=-(n-1)\cos^{n-2}x\sin x\,dx$ and $v=\sin x$

$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx$

$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)dx$

$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$

$\Rightarrow \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

b. Let $n=2$ in (a) $\longrightarrow \int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx$

$= \frac{1}{4}\sin(2x) + \frac{x}{2} + C$

c. $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx$

$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{16}\sin(2x) + \frac{3x}{8} + C$

48. Find the area of the region bounded by $y = 5\ln x$ and $y = x\ln x$.

The curves intersect when $(x-5)\ln x = 0$, that is when $x = 1$ or $x = 5$. For $1 < x < 5$ we have $5\ln x > x\ln x$, since $\ln x > 0$.

$$\rightarrow \text{Area} = \int_1^5 (5\ln x - x\ln x)\,dx$$

Let $u = \ln x$ $\quad dv = (5-x)\,dx$

$\rightarrow du = \frac{dx}{x}$ $\quad v = 5x - \frac{1}{2}x^2$

$$\rightarrow \text{Area} = (\ln x)\left[5x - \tfrac{1}{2}x^2\right]_1^5 - \int_1^5 \left(5x - \tfrac{1}{2}x^2\right)\frac{dx}{x}$$

$$= (\ln 5)\left(\tfrac{25}{2}\right) - 0 - \int_1^5 \left(5 - \tfrac{1}{2}x\right)dx$$

$$= \tfrac{25}{2}\ln 5 - \left[5x - \tfrac{1}{4}x^2\right]_1^5 = \tfrac{25}{2}\ln 5 - 14$$