SECTION 17.2 # 2, 6, 8, 16

(2) $y''+9y=e^{3x}$

FIRST FIND A GENERAL SOLUTION OF $y''+9y=0$

$r^2+9=0 \Rightarrow r=\pm 3i$

$y_c = c_1\cos 3x + c_2\sin 3x$ IS A GENERAL SOLUTION

FOR y_p WE "GUESS" Ae^{3x} SINCE OUR EQUATION IS OF THE FORM $ay''+by'+cy=G(x)$ WHERE $G(x)=e^{3x}$.

$$\left.\begin{aligned} y_p &= Ae^{3x} \\ y'_p &= 3Ae^{3x} \\ y''_p &= 9Ae^{3x} \end{aligned}\right\}$$ SUBSTITUTE INTO ORIGINAL EQUATION TO GET

$$9Ae^{3x}+9Ae^{3x}=e^{3x}$$
$$18A\cdot e^{3x}=e^{3x}$$
$$18A=1$$
$$A=1/18$$
SO $y_p=\frac{1}{18}e^{3x}$

BY THEOREM $\boxed{3}$ $y=y_p+y_c$ IS A GENERAL SOLUTION OF OUR EQUATION, SO

$$\boxed{y=\frac{1}{18}e^{3x}+c_1\cos 3x+c_2\sin 3x}$$

(6) $y''+2y'+y=xe^{-x}$

FIRST FIND A GENERAL SOLUTION OF $y''+2y'+y=0$ (COMPLEMENTARY EQUATION)

$r^2+2r+1=0$

$(r+1)^2=0$ SO $r=-1$

$y_c=c_1e^{-x}+c_2xe^{-x}$

FOR y_p TRY $x^2(Ax+B)e^{-x}$ (SO THAT NONE OF THE TERMS IN y_p IS A SOLUTION OF THE COMPLEMENTARY EQUATION. NOTICE THAT SIMPLY $(Ax+B)e^{-x}$ WOULDN'T DO SINCE ANYTHING OF THAT FORM IS A SOLUTION OF THE COMPLEMENTARY EQUATION.

SIMILARLY, $x(Ax+B)e^{-x}$ WOULD HAVE A TERM THAT IS A SOLUTION

OF THE COMPLEMENTARY EQUATION, BUT $x^2(Ax+B)e^{-x}$ WILL DO SINCE NONE OF ITS TERMS IS A SOLUTION OF THE COMPLEMENTARY EQUATION).

$$y_p = x^2(Ax+B)e^{-x} = \underline{(Ax^3+Bx^2)e^{-x}}$$
$$y_p' = e^{-x}(3Ax^2+2Bx) - e^{-x}(Ax^3+Bx^2) = \underline{e^{-x}(x^3(-A)+x^2(3A-B)+2Bx)}$$
$$y_p'' = e^{-x}(-3Ax^2+(6A-2B)x+2B) - e^{-x}(-Ax^3+(3A-B)x^2+2Bx) =$$
$$= \underline{e^{-x}(Ax^3+(B-6A)x^2+(6A-4B)x+2B)}$$

SUBSTITUTING THESE INTO OUR EQUATION GIVES:

$$y''+2y'+y = e^{-x}\left[x^3(A-2A+A)+x^2(-6A+B+6A-2B+B)+\right.$$
$$\left.+x(6A-4B+4B)+2B\right] =$$
$$= e^{-x}(6Ax+2B)$$
$$y''+2y'+y = xe^{-x}$$

$$e^{-x}(6Ax+2B) = xe^{-x}$$
$$6Ax+2B = x \quad \text{SO } \underline{B=0} \text{ AND } 6A=1$$
$$\underline{A = 1/6}$$

$$y_p = x^2\cdot\frac{1}{6}xe^{-x} = \frac{1}{6}x^3e^{-x}$$

$$\boxed{y = y_p+y_c = \frac{1}{6}x^3e^{-x} + c_1e^{-x} + c_2xe^{-x}}$$

8 $y''-4y = e^x\cos x$ INITIAL CONDITIONS: $y(0)=1$, $y'(0)=2$

$$r^2-4=0 \qquad r=\pm 2$$
$$y_c = c_1e^{2x}+c_2e^{-2x}$$

FOR y_p GUESS $Ae^x\cos x + Be^x\sin x = e^x(A\cos x + B\sin x)$.

THIS IS A REASONABLE GUESS BECAUSE NONE OF ITS TERMS SOLVES

THE COMPLEMENTARY EQUATION, AND IT IS OF THE FORM "$e^x \cos x$". WE ALSO "THROW IN" THE $Be^x \sin x$ TERM TO TAKE INTO ACCOUNT DERIVATIVES OF $\sin x$ AND $\cos x$.

$$y_p = e^x\,(A\cos x + B\sin x)$$
$$y_p' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$
$$y_p'' = e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x) + e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x) =$$
$$= e^x\left[(\cos x)(-A+B+B+A) + (\sin x)(-B-A-A+B)\right] =$$
$$= e^x\,(2B\cos x - 2A\sin x)$$

SUBSTITUTION GIVES

$$y' - 4y = e^x(2B\cos x - 2A\sin x - 4A\cos x - 4B\sin x) =$$
$$= e^x\left[(2B-4A)\cdot\cos x + (-4B-2A)\sin x\right]$$

$y'' - 4y = e^x \cos x$ SO

$$-2A - 4B = 0 \quad \ldots\ldots (1)$$
$$-4A + 2B = 1 \quad \ldots\ldots (2)$$

(1) + 2(2) GIVES

$-10A = 2$ OR $\boxed{A = \frac{1}{5}}$

$B = \frac{1+4A}{2} = \frac{1+\frac{4}{5}}{2} = \boxed{\frac{1}{10}}$

$$y_p = e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$$

$$\boxed{y = y_p + y_c = e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right) + c_1e^{2x} + c_2e^{-2x}}$$

THIS IS A GENERAL SOLUTION OF $y'' - 4y = e^x\cos x$. TO GET A PARTICULAR SOLUTION, WE USE THE INITIAL CONDITIONS.

(I) $y(0) = 1$

$y(0) = -\frac{1}{5} + c_1 + c_2$ SO $c_1 + c_2 - \frac{1}{5} = 1$

(II) $y'(0)=2$

$$y' = 2c_1e^{2x} - 2c_2e^{-2x} + e^x\left(\frac{1}{10}\sin x - \frac{1}{5}\cos x\right) + e^x\left(\frac{1}{10}\cos x + \frac{1}{5}\sin x\right)$$

$$y'(0) = 2c_1 - 2c_2 - \frac{1}{5} + \frac{1}{10}$$

$$y'(0) = 2c_1 - 2c_2 - \frac{1}{10}$$

$$2c_1 - 2c_2 - \frac{1}{10} = 2$$

WE SOLVE THE SYSTEM OF EQUATIONS

$$c_1 + c_2 - 1/5 = 1$$
$$2c_1 - 2c_2 - 1/10 = 2$$

$$c_1 + c_2 = 6/5 \quad \ldots\ldots (1)$$
$$2c_1 - 2c_2 = 21/10 \quad \ldots (2)$$

$2\cdot(1) - (2)$ GIVES

$$4c_2 = \frac{3}{10}$$

$$\boxed{c_2 = 3/40}$$

$$c_1 = \frac{6}{5} - \frac{3}{40} = \frac{45}{40} = \boxed{\frac{9}{8}}$$

PARTICULAR SOLUTION:

$$\boxed{y = e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right) + \frac{9}{8}e^{2x} + \frac{3}{40}e^{-2x}}$$

(16) $y'' + 3y' = 1 + xe^{-3x}$

$r^2 + 3r = 0$

$r(r+3) = 0$ SO $r=0$ OR $r=-3$

$y_c = c_1 + c_2e^{-3x}$ → COMPLEMENTARY EQUATION

FOR $y'' + 3y' = 1$ WE GUESS $\boxed{y_{p_1} = Ax}$ (ONLY A WOULD SOLVE y_c)

FOR $y'' + 3y' = xe^{-3x}$ WE GUESS $\boxed{(Bx+C)xe^{-3x}}$ (ONLY Bxe^{-3x} WOULD SOLVE y_c)