11.6 SOLUTIONS.

6. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{3^n}{n!}$

$\Rightarrow$ Use ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{3 \times \cancel{3^n}}{\cancel{n!}\,(n+1)} \right) \times \left(\frac{\cancel{n!}}{\cancel{3^n}} \right) = \left(\frac{3}{n+1} \right) \Longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow$ We can conclude absolut convergence.

10. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1} \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$\Rightarrow$ Use Limit Comparison Test with $\sum\left(\frac{1}{n}\right)$, a divergent series (Harmonic)

$$\lim_{n\to\infty} \left(\frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{1}{n}\right)} \right) = \lim_{n\to\infty} \left(\frac{n^2}{n^2+1} \right) = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n^2}} \right) = 1$$

$\Rightarrow \sum (-1)^{n-1} \frac{n}{n^2+1}$ is not absolutelly convergent.

Is it convergent conditionally?

$\Rightarrow 1^{st}$ Condition: $|a_{n+1}| \leqslant |a_n|$

$$\Rightarrow \frac{n}{n^2+1} \geqslant \frac{n+1}{n^2+2n+2}$$

$$\Rightarrow \cancel{n^3} + 2n^2 + 2n \geqslant \cancel{n^3} + n^2 + n + 1$$

$$\Rightarrow n^2 + n \geqslant 1 \quad \text{for all } n \geqslant 1$$

$\therefore 1^{st}$ Condition satisfied!

10. cont'd

2nd Condition: $\lim_{n\to\infty} |a_n| = 0$

$$\Rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1}{n+\frac{1}{n}}\right) = 0$$

: 2nd Condition satisfied !

$\Rightarrow$ We can conclude Conditional Convergence.

12. Optional. $\sum_{n=1}^{\infty} e^{-n} n! \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} e^{-n} n!$

$\Rightarrow$ Use ratio test:

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \frac{e^{-(n+1)}\,(n+1)\cancel{n!}}{e^{-n}\,(\cancel{n!})} = \frac{\cancel{e^{-n}}\,(n+1)}{e\cdot \cancel{e^{-n}}} = \frac{n+1}{e} \longrightarrow \infty \text{ as } n\to\infty$$

$\Rightarrow \sum_{n=1}^{\infty} e^{-n} * n!$ diverges

18. Optional. $\sum_{n=1}^{\infty} \left(\cos(n\pi/6)\right)/n^{3/2}$ $\Rightarrow$ Use comparison test with $\sum \frac{1}{n^{3/2}}$

$\Rightarrow \sum \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$, thus converges.

$\Rightarrow$ Claim: $\frac{\cos(n\pi/6)}{n^{3/2}} \overset{?}{\leq} \frac{1}{n^{3/2}}$

$\Rightarrow \cos(n\pi/6) \overset{?}{\leq} 1$ $\Rightarrow$ this is always the case since $\cos(x)$ oscillates between -1 and 1

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\left(\cos(\pi n/6)\right)}{n^{3/2}} \text{ Converges}$$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n} \Rightarrow \sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$: Use Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n)^n}{\left(\ln(n+1)\right)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(\ln n)^n}^{\,1}}{\cancel{\ln(n+1)^n}\times \cancel{\ln(n+1)}^{\,\infty}}\right| = 0$$

$\Rightarrow$ Conclude Absolute Convergence.

OR: Root Test: $\displaystyle\lim_{n\to\infty}\left(\sqrt[n]{a_n}\right) = \lim_{n\to\infty}\left(\sqrt[n]{(\ln n)^{-n}}\right) = \lim_{n\to\infty}\left(\frac{1}{\ln n}\right) = 0 < 1$: Absolute Convergence

28.
Optional.

$$\frac{1}{3} + \frac{1\times4}{3\times5} + \frac{1\times4\times7}{3\times5\times7} + \frac{1\times4\times7\times10}{3\times5\times7\times9} + \dots + \frac{1\times4\times\dots(3n-2)}{3\times5\times\dots(2n+1)}$$

$\Rightarrow$ Notice that the numerator grows at a faster rate than the denomi for for $n \geqslant 3$ $(3n-2 > 2n+1$ for $n \geqslant 3)$.

Thus the terms in the series will diverge and the series itself will diverge (Divergence Test)

30.

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5\times8\times11\times\dots(3n+2)} \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2^n n!}{5\times8\times11\times\dots(3n+2)}$$

$\Rightarrow$ Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{2\cdot\cancel{2^n}\times\cancel{(1\cdot2\cdot\dots\cdot n)}(n+1)}{\cancel{5\times8\times11\times\dots}\times(3n+8)} \times \frac{\cancel{5\times8\times11\times\dots}\times(3n+2)}{\cancel{2^n}\cancel{(n!)}}\right)$$

$$= \left(\frac{2\times(n+1)}{3n+8}\right) = \left(\frac{2n+2}{3n+8}\right) \longrightarrow \left(\frac{2}{3}\right) \text{ as } n\to\infty$$

$\Rightarrow \displaystyle\sum (-1)^n \frac{2^n n!}{5\times8\times11\times\dots\times(3n+2)}$ is Absolutely Convergent.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$, find k for which series converges.

$\Rightarrow$ Use <u>Ratio Test</u> and make $L < 1$.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{(n+1)^2 \cancel{(n!)^2}}{\cancel{(kn)!}\,(kn+1)\ldots(kn+k)} \times \frac{\cancel{(kn)!}}{\cancel{(n!)^2}} \right) = \left(\frac{(n+1)(n+1)}{(kn+1)(kn+2)\ldots(kn+k)} \right) <$$

$\Rightarrow$ holds true if $k \geq 2$

$$\Rightarrow k=1 \ : \ L = \lim \left(\frac{(n+1)^2}{(n+1)} \right) = \lim_{n\to\infty} (n+1) = \infty$$

$$\Rightarrow k=2 \ : \ L = \lim_{n\to\infty} \left(\frac{(n+1)(n+1)}{(2n+1)(2n+2)} \right) = 1/4 < 1$$

$$\Rightarrow k \geq 3 \ : \ L = \lim_{n\to\infty} \left(\frac{(n+1)^2}{(kn+1)(kn+2)\ldots(kn+k)} \right) = 0 < 1$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ converges only for $k \geq 2$