11.5 SOLUTIONS

6. $\sum_{n=1}^{\infty} (-1)^{n-1} / (3n-1) = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

$\Rightarrow$ 1st Condition for convergence: $|b_{n+1}| \leq |b_n|$

$: \frac{1}{3n-1} \overset{?}{\geqslant} \frac{1}{3(n+1)-1}$

$3n+2 \overset{?}{\geqslant} 3n-1$

$2 \geqslant -1 \Rightarrow$ Always true : 1st Condition satisfied

$\Rightarrow$ 2nd Condition : $\lim(b_n) = 0$?

$: \lim\left(\frac{1}{3n-1}\right) = 0$: 2nd Condition satisfied

$\Rightarrow$ Series converges

10. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2}{4n^2+1} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

$\Rightarrow$ 2nd Condition first: $\lim(b_n) = 0$?

$: \lim_{n\to\infty}\left(\frac{2n^2}{4n^2+1}\right) = \lim_{n\to\infty}\left(\frac{2}{4+\frac{1}{n^2}}\right) = \frac{1}{2} \neq 0$: 2nd Condition not satisfie

Alternating Series Test fails!

$\Rightarrow$ Use divergence test for divergence

$\lim_{n\to\infty}\left((-1)^{n-1} \frac{2n^2}{4n^2+1}\right) \neq 0$ (approaches ½ in magnitude, but is not define since terms alternate in signs)

$\Rightarrow$ We have to conclude divergence

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

$\Rightarrow$ 1st Condition: $\frac{\ln(n+1)}{n+1} \overset{?}{\leq} \frac{\ln(n)}{n}$

: look at the derivative of $f(n) = \frac{\ln n}{n}$ and if it's negative for some n large enough until infinity, then $f(n)$ decreases beyond that n.

$f(n) = \frac{\ln n}{n} \Rightarrow f'(n) = \frac{1 - \ln(n)}{n^2}$

$\Rightarrow f'(n) = \frac{1 - \ln n}{n^2} \overset{?}{<} 0$

$1 \overset{?}{>} \ln(n)$: true for $n \geq 3$

$\Rightarrow$ terms of the series will be decreasing beyond $n=2$, so the first condition is satisfied (Rember you can add a finite number to a converging series and the series will still be converging)

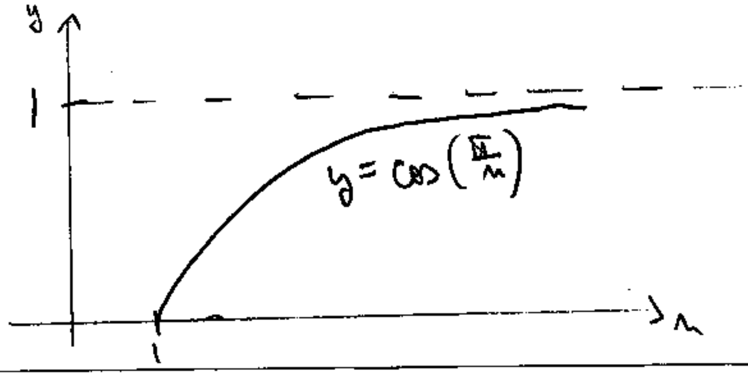
$\Rightarrow$ 2nd Condition : $\lim(b_n) = 0$?

: $\lim_{n\to\infty}(b_n) = \lim_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = 0$ (L'Hôpital's)

18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right) = \sum_{n=1}^{\infty} (-1)^n b_n$

$\Rightarrow$ 2nd Condition first: $\lim(b_n) = 0$?

: $\lim\left(\cos\frac{\pi}{n}\right) = 1 \neq 0$



$\Rightarrow$ 2nd Condition not satisfied: Alternating Series test fails

: Use Divergence test to show divergence:

$$\lim_{n\to\infty} \left((-1)^{n-1} \cos\left(\frac{\pi}{n}\right)\right) \neq 0$$ (approaches 1 in magnitude, but is undefined since the series alternates in signs)

35. Optional. $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, for $b_n \begin{cases} 1/n, & \text{for odd } n \\ 1/n^2, & \text{otherwise} \end{cases}$

$$\left\{1, -\tfrac{1}{4}, \tfrac{1}{3}, -\tfrac{1}{16}, \tfrac{1}{5}, -\tfrac{1}{36}, \tfrac{1}{7} \ldots\right\}$$

$\Rightarrow$ 1st Condition for convergence: $b_n \geq b_{n+1}$

i) n is odd : $\frac{1}{n} \overset{?}{\geq} \frac{1}{(n+1)^2}$

$(n+1)^2 \geq n \Rightarrow$ true for all n.

ii) n is even : $b_n \overset{?}{\geq} b_{n+1}$

$$\frac{1}{n^2} \geq \frac{1}{n+1}$$

$n+1 \geq n^2$: not true for $n \geq 2$

$\Rightarrow$ We cannot apply the Alternating Series Test because there are 2 cases we have to consider and while the first condition holds in one case, it doesn't hold in the other.

36. a) $S_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2N}$

$h_{2N} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2N} = \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2N-1}\right) + \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2N}\right)$

$h_N = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}$

$\Rightarrow h_{2N} - h_N = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2N}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{N}\right) =$

$= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2N-1}\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2N}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}\right)$

$= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2N-1}\right) + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{2N}\right)$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{2N-1} - \frac{1}{2N} = S_{2N}$ As required

b) $h_{2N} - \ln(2n) \to \gamma$ as $n \to \infty$

$h_N - \ln(n) \to \gamma$ as $n \to \infty$

$\Rightarrow S_{2N} = h_{2N} - h_N = \overset{\gamma}{\left[h_{2N} - \ln(2n)\right]} - \overset{\gamma}{\left[h_N - \ln(n)\right]} + \ln(2n) - \ln(n) \Rightarrow$

$\to \gamma - \gamma + \ln(2n) - \ln(n) \Rightarrow \ln(2n) - \ln(n) \to \underline{\ln(2)}$ as $n \to \infty$

As required